Star products: a group-theoretical point of view

Paolo Aniello

Dipartimento di Scienze Fisiche dell'Università di Napoli 'Federico II',
Complesso Universitario di Monte S. Angelo, via Cintia, 80126 Napoli, Italy
and
Istituto Nazionale di Fisica Nucleare (INFN) – Sezione di Napoli
and
Facoltà di Scienze Biotecnologiche, Università di Napoli 'Federico II'
E-mail: paolo.aniello@na.infn.it

April 7, 2009

Abstract

Adopting a purely group-theoretical point of view, we consider the star product of functions which is associated, in a natural way, with a square integrable (in general, projective) representation of a locally compact group. Next, we show that for this (implicitly defined) star product explicit formulae can be provided. Two significant examples are studied in detail: the group of translations on phase space and the one-dimensional affine group. The study of the first example leads to the Grönewold-Moyal star product. In the second example, the link with wavelet analysis is clarified.

1 Introduction

The concept of star product of functions is a remarkable achievement of theoretical physics. The archetype — and still nowadays, the most important realization — of this concept is the Grönewold-Moyal star product (see [1] and references therein). Although there is no unique general mathematical framework encompassing all known star products, one can certainly single out a simple leading idea to which the various possible definitions of star products are more or less inspired: to replace the ordinary pointwise product of ($\mathbb{C}$-valued) functions defined on a certain set (a 'phase space' endowed with some structures: a differentiable manifold, a measure space etc.) with a suitable non-commutative, associative product that mimics the typical non-commutative behavior of linear operators.

We will make no attempt at surveying the rich and varied literature on star products. We will content ourselves with recalling that both differential-geometric [3, 4, 5] and algebraic [6] approaches to the subject have been adopted, also in view of different purposes and applications. It is also worth mentioning the fact that the most important topics where the formalism of star products plays a relevant role are, probably, the construction of quantum mechanics 'on phase space' and the study of the classical limit of quantum mechanics [1, 7]. Thus, one may regard E. Wigner [8] and H. Weyl [9] as the fathers of this formalism.

More recently, a general approach to star products based on the idea of using suitable 'quantizers' and 'dequantizers' has been proposed and developed by various authors [10, 11, 12, 13, 14, 15, 16]. This approach is very close to applications in quantum mechanics since the star products of functions that one obtains are, by construction, nothing but the 'images' of the products of quantum-mechanical operators.

In our present contribution, we will adopt a purely group-theoretical point of view which is conceptually similar to the 'quantizer-dequantizer' approach cited above. Indeed, rather than trying to define a star product directly in a given space of functions (as usual, for instance, in the differential-geometric approach), we consider the star product (implicitly) induced by a suitable group-theoretical quantization-dequantization scheme. Clearly, at this point, the real problem is to find explicit formulae for the implicitly defined star product.

Before illustrating the main points of our work, it is worth mentioning that recently another group-theoretical approach to star products — in the context of a suitable quantization-dequantization scheme — has been elaborated; see ref. [17]. However, this approach, differently from the approach adopted in the present paper, relies on the concept of 'frame transform' and it is not directly related to the Groenewold-Moyal product.

Let us now briefly outline our method and our main results. First, we show that by means of the quantization (Weyl) and dequantization (Wigner) maps generated by a square integrable (in general, projective) representation U of a locally compact group G — see [17, 18, 19] — it is possible to introduce, in a natural way, a star product in the Hilbert space $\mathrm{L}^2(G)$ of square integrable $\mathbb{C}$-valued functions on G. The product of two functions is obtained by quantizing them, by forming the product of the two operators thus obtained and, finally, by dequantizing this product. Endowed with the operation just described, $\mathrm{L}^2(G)$ becomes a H*-algebra. We will then prove — this is the main result of the paper — that the star product in $\mathrm{L}^2(G)$ admits a simple explicit formula. More precisely, we will show that with every orthonormal basis in the Hilbert space of the representation U is associated a formula for the star product (however, all these formulae share the same general form). This basic result can be generalized or specialized in various ways. For instance, an expression of the '$\hat{K}$-deformed star product' — see [13, 14] — which is an interesting generalization of the star product, can also be obtained. On the other hand, in the case where G is unimodular, a particularly simple formula for the star product — a sort of 'twisted convolution' *à la* Grossmann-Loupias-Stein [20] — can be derived.

We believe that the point of view on star products adopted in this paper is very close to the 'original spirit' of the Grönewold-Moyal star product since it solely relies on (generalized) Wigner and Weyl maps. In fact, 'our' star product is *essentially* the Grönewold-Moyal star product in the case where the group G is the group of translations on phase space; i.e., the two products — the twisted convolution and the Grönewold-Moyal product — are related by the symplectic Fourier transform.

We stress that our approach relies on the *existence* of a square integrable representation U of the locally compact group G for defining an associated star product in $\mathrm{L}^2(G)$. This feature, however, should not be regarded as a limit of this approach. As is well known, when dealing with mathematics nothing is free: the weaker are the assumptions, the poorer will be the results that one is able to prove. Moreover, our group-theoretical point of view is very natural having in mind applications to physics. If G is regarded as a 'symmetry group' of a quantum system and U as the symmetry action of this group in the Hilbert space $\mathcal{H}$ of the system, then the associated star product in $\mathrm{L}^2(G)$ is nothing but the realization in terms of functions of the product of quantum-mechanical operators (observables or states); moreover, it turns out that the star product is 'equivariant' with respect to the natural action of the symmetry group. Namely, the natural action of G on operators in $\mathcal{H}$ translates into (i.e. is intertwined by the dequantization map with) a simple transformation of the corresponding functions in $\mathrm{L}^2(G)$, and the star product of two *transformed* functions coincides with the *transformed* product of the two *untransformed* functions.

The paper is organized as follows. In Sect. 2, we fix the main notations and we briefly recall some mathematical notions; in particular, we review some basic facts concerning square integrable representations. Next, in Sect. 3, we define the dequantization (Wigner) and quanti-

zation (Weyl) maps generated by a square integrable representation, and we derive the relevant 'intertwining properties' of the Wigner map. On the basis of these definitions we then introduce — see Sect. 4 — the notion of star product associated with a square integrable representation, and we study its main properties. The star product introduced in such a way is, however, only *implicitly* defined. As already mentioned, it is a remarkable fact that it admits an *explicit* realization; furthermore, in the case of a unimodular group, a particularly simple formula can be derived. These results — that form the *core* of our paper — are stated and proved in Sect. 5. In Sect. 6, we consider two significant examples: the group of translations on phase space — which is related to the standard Grönewold-Moyal star product — and the affine group, which plays a central role in wavelet analysis. Eventually, in Sect. 7, a few conclusions are drawn, with a glance at various possible developments of our work.

2 Some known facts and notations

In this section, we will recall some basic facts of the theory of representations of topological groups; standard references on the subject are [21, 22]. We will also fix the main notations that will be used in the following sections.

Let G be a locally compact, second countable, Hausdorff topological group (in short, l.c.s.c. group). We will denote by μ_G and Δ_G, respectively, a *left Haar measure* (of course uniquely defined up to a multiplicative constant) and the *modular function* on G. The symbol e will indicate the unit element in G.

For the scalar product $\langle \cdot, \cdot \rangle$ in a separable complex Hilbert space $\mathcal{H}$, we will always follow the convention that it is linear in the *second* argument. The symbol $\mathcal{U}(\mathcal{H})$ will denote the *unitary group* of $\mathcal{H}$ — i.e. the group of all unitary operators in $\mathcal{H}$, endowed with the strong operator topology — which is a metrizable, second countable, Hausdorff topological group.

We will mean by the term *projective representation* of a l.c.s.c. group G a Borel projective representation of G in a separable complex Hilbert space $\mathcal{H}$ (see, for instance, ref. [21], chapter VII), namely a map of G into $\mathcal{U}(\mathcal{H})$ such that

1. U is a weakly Borel map, i.e. $G \ni g \mapsto \langle \phi, U(g)\, \psi \rangle \in \mathbb{C}$ is a Borel function,[1] for any pair of vectors $\phi, \psi \in \mathcal{H}$;

2. $U(e) = I$, where I is the identity operator in $\mathcal{H}$;

3. denoting by $\mathbb{T}$ the circle group, namely the group of complex numbers of modulus one, there exists a Borel function $\mathtt{m} : G \times G \to \mathbb{T}$ such that

$$U(gh) = \mathtt{m}\,(g,h)\, U(g)\, U(h), \quad \forall\, g,h \in G. \tag{1}$$

The function $\mathtt{m}$ — which is called the *multiplier associated with U* — satisfies the following conditions:

$$\mathtt{m}\,(g,e) = \mathtt{m}\,(e,g) = 1, \quad \forall\, g \in G, \tag{2}$$

and

$$\mathtt{m}\,(g_1, g_2 g_3)\, \mathtt{m}\,(g_2, g_3) = \mathtt{m}\,(g_1 g_2, g_3)\, \mathtt{m}\,(g_1, g_2), \quad \forall\, g_1, g_2, g_3 \in G. \tag{3}$$

It is, moreover, immediate to check that $\mathtt{m}\,(g, g^{-1}) = \mathtt{m}\,(g^{-1}, g)$. Clearly, in the case where $\mathtt{m} \equiv 1$, U is a standard unitary representation; in this case, according to a well known result, the hypothesis that the map U is weakly Borel implies that it is, actually, strongly continuous.

[1] The terms *Borel function* (or map) and *Borel measure* will be always used with reference to the natural Borel structures on the topological spaces involved, namely to the smallest σ-algebras containing all open subsets.

The notion of irreducibility is defined for projective representations as for standard unitary representations.

Let $\widetilde{U}\colon G \to \mathcal{U}(\widetilde{\mathcal{H}})$ be a projective representation of G in a (separable complex) Hilbert space $\widetilde{\mathcal{H}}$. We say that $\widetilde{U}$ is *physically equivalent* to U if there exist a Borel function $\beta\colon G \to \mathbb{T}$, and a unitary or antiunitary operator $W\colon \mathcal{H} \to \widetilde{\mathcal{H}}$, such that

$$\widetilde{U}(g) = \beta(g)\, W\, U(g)\, W^*, \quad \forall g \in G. \tag{4}$$

It is obvious that a projective representation — physically equivalent to an irreducible projective representation — is irreducible too. We will say that the representations U and $\widetilde{U}$ are *unitarily equivalent* if, in relation (4), $\beta \equiv 1$ and W is a unitary operator.

Observe that we can identify the unitary dual of G with any (suitably topologized) maximal set of mutually unitarily inequivalent, irreducible, unitary representations of G. We will denote by $\check{G}$ such a set, and we will call it *a realization of the unitary dual of* G. It is well known that, if G is compact,[2] then $\check{G}$ is a finite or countable set (endowed with the discrete topology); moreover, $\check{G}$ consists of finite-dimensional representations.

Let U be an *irreducible* projective representation of the l.c.s.c. group G in the Hilbert space $\mathcal{H}$. Then, given two vectors $\psi, \phi \in \mathcal{H}$, we define the function (usually called 'coefficient' of the representation U)

$$\mathsf{c}^U_{\psi,\phi}\colon G \ni g \mapsto \langle U(g)\,\psi, \phi\rangle \in \mathbb{C}, \tag{5}$$

and we consider the set (of 'admissible vectors for U')

$$\mathsf{A}(U) := \left\{\psi \in \mathcal{H} \mid \exists \phi \in \mathcal{H}:\ \phi \neq 0,\ \mathsf{c}^U_{\psi,\phi} \in \mathrm{L}^2(G)\right\}, \tag{6}$$

where $\mathrm{L}^2(G) \equiv \mathrm{L}^2(G, \mu_G; \mathbb{C})$ (in the following, we will denote by $\langle\cdot,\cdot\rangle_{\mathrm{L}^2}$ and $\|\cdot\|_{\mathrm{L}^2}$ the scalar product and the norm in $\mathrm{L}^2(G)$). The representation U is said to be *square integrable* if $\mathsf{A}(U) \neq \{0\}$. Square integrable projective representations are characterized by the following result — see ref. [23] — which is a generalization of a classical theorem of Duflo and Moore [24] concerning unitary representations:

Theorem 1 *Let the projective representation $U\colon\ G \to \mathcal{U}(\mathcal{H})$ be square integrable. Then, the set $\mathsf{A}(U)$ is a dense linear span*[3] *in $\mathcal{H}$, stable under the action of U, and, for any pair of vectors $\phi \in \mathcal{H}$ and $\psi \in \mathsf{A}(U)$, the coefficient $\mathsf{c}^U_{\psi,\phi}$ is square integrable with respect to the left Haar measure μ_G on G. Moreover, there exists a unique positive selfadjoint, injective linear operator $\hat{D}_U$ in $\mathcal{H}$ —* which we will call the 'Duflo-Moore operator associated with U' *— such that*

$$\mathsf{A}(U) = \mathrm{Dom}\big(\hat{D}_U\big) \tag{7}$$

and the following 'orthogonality relations' hold:

$$\big\langle \mathsf{c}^U_{\psi_1,\phi_1}, \mathsf{c}^U_{\psi_2,\phi_2}\big\rangle_{\mathrm{L}^2} = \langle\phi_1,\phi_2\rangle\,\big\langle \hat{D}_U\,\psi_2, \hat{D}_U\,\psi_1\big\rangle, \tag{8}$$

for all $\phi_1,\phi_2 \in \mathcal{H}$ and all $\psi_1,\psi_2 \in \mathsf{A}(U)$. The Duflo-Moore operator $\hat{D}_U$ is semi-invariant — with respect to U — with weight $\Delta_G^{1/2}$, i.e.

$$U(g)\,\hat{D}_U = \Delta_G(g)^{\frac{1}{2}}\,\hat{D}_U\,U(g), \quad \forall g \in G; \tag{9}$$

it is bounded if and only if G is unimodular (i.e. $\Delta_G \equiv 1$) *and, in such case, it is a multiple of the identity.*

---

[2] We will include among the compact groups all the finite groups (endowed with the discrete topology).

[3] Throughout the paper, we call a nonempty subset of a vector space V a 'linear span' if it is a linear space itself (with respect to the operations of V), *with no extra requirement of closedness with respect to any topology on* V; we prefer to use the term '(vector) subspace' of V for indicating a *closed* linear span (with respect to a given topology on V).

Remark 1 If U is square integrable, the associated Duflo-Moore operator $\hat{D}_U$, being injective and positive selfadjoint, has a positive selfadjoint densely defined inverse. In the case where U is a unitary representation, Duflo and Moore call the square of $\hat{D}_U^{-1}$ the *formal degree* of the representation U. Note that the operator $\hat{D}_U$ is linked to the normalization of the Haar measure μ_G: if μ_G is rescaled by a positive constant, then $\hat{D}_U$ is rescaled by the square root of the same constant. We will say, then, that $\hat{D}_U$ is *normalized according to* μ_G. On the other hand, if a normalization of the left Haar measure on G is not fixed, $\hat{D}_U$ is defined up to a positive factor and we will call a specific choice a *normalization of the Duflo-Moore operator*. In particular, if G is unimodular, then $\hat{D}_U = I$ is a normalization of the Duflo-Moore operator; the corresponding Haar measure will be said to be *normalized in agreement with the representation* U. Moreover, observe that, as a consequence of relation (9), the dense linear span $\mathrm{Dom}\big(\hat{D}_U^{-1}\big) = \mathrm{Ran}\big(\hat{D}_U\big)$ — like the linear span $\mathsf{A}(U) = \mathrm{Dom}\big(\hat{D}_U\big)$ — is stable under the action of U and

$$U(g)^{-1}\,\hat{D}_U^{-1} = \Delta_G(g)^{\frac{1}{2}}\,\hat{D}_U^{-1}\,U(g)^{-1}, \quad \forall\, g \in G. \tag{10}$$

From this relation, using the fact that $U(g)^{-1} = \mathtt{m}\,(g, g^{-1})\,U(g^{-1})$, we obtain:

$$U(g)\,\hat{D}_U^{-1} = \Delta_G(g)^{-\frac{1}{2}}\,\hat{D}_U^{-1}\,U(g), \quad \forall g \in G, \tag{11}$$

We finally note that, in the case where G is not unimodular, a square integrable representation of G cannot be finite-dimensional (since the associated Duflo-Moore operator is unbounded). ■

Let us list a few basic facts about square integrable representations:

- The square-integrability of a representation is a property which extends to all its physical equivalence class.

- If the representation U of G is square integrable, then the orthogonality relations (8) imply that, for every nonzero admissible vector $\psi \in \mathsf{A}(U)$, one can define the linear operator

$$\mathfrak{W}_U^{\psi}\colon \mathcal{H} \ni \phi \mapsto \big\|\hat{D}_U\,\psi\big\|^{-1}\,\mathsf{c}_{\psi,\phi}^{U} \in \mathrm{L}^2(G) \tag{12}$$

  — sometimes called *(generalized) wavelet transform* generated by U, with *analyzing* or *fiducial vector* ψ — which is an isometry. For the adjoint $\mathfrak{W}_U^{\psi\,*}\colon \mathrm{L}^2(G) \to \mathcal{H}$ of the isometry $\mathfrak{W}_U^{\psi}$ the following weak integral 'reconstruction formula' holds:

$$\mathfrak{W}_U^{\psi\,*}\,f = \big\|\hat{D}_U\,\psi\big\|^{-1}\int_G f(g)\,\big(U(g)\,\psi\big)\,\mathrm{d}\mu_G(g), \quad \forall f \in \mathrm{L}^2(G). \tag{13}$$

  The ordinary wavelet transform arises in the special case where G is the 1-dimensional affine group $\mathbb{R} \rtimes \mathbb{R}_*^+$ (see [25, 26]); we will better clarify this point in Sect. 6.

- The isometry $\mathfrak{W}_U^{\psi}$ intertwines the square integrable representation U with the *left regular* $\mathtt{m}$*-representation* $R_{\mathtt{m}}$ of G in $\mathrm{L}^2(G)$, see ref. [23], which is the projective representation (with multiplier $\mathtt{m}$) defined by:

$$\big(R_{\mathtt{m}}(g)f\big)(g') = \vec{\mathtt{m}}\,(g, g')\,f(g^{-1}g'), \quad g, g' \in G, \tag{14}$$

$$\vec{\mathtt{m}}\,(g, g') := \mathtt{m}\,(g, g^{-1})^*\,\mathtt{m}\,(g^{-1}, g'), \tag{15}$$

  for every $f \in \mathrm{L}^2(G)$; namely:

$$\mathfrak{W}_U^{\psi}\,U(g) = R_{\mathtt{m}}(g)\,\mathfrak{W}_U^{\psi}, \quad \forall g \in G. \tag{16}$$

  Hence, U is (unitarily) equivalent to a subrepresentation of $R_{\mathtt{m}}$. Note that, for $\mathtt{m} \equiv 1$, $R \equiv R_{\mathtt{m}}$ is the standard left regular representation of G.

- Let the group G be compact (hence, unimodular), and let $\check{G}$ be a realization of the unitary dual of G. In this case, the *unitary* irreducible representations of G are all finite-dimensional — we will denote by $\delta(U)$ the dimension of the Hilbert space of the representation $U \in \check{G}$ — and square integrable (since the Haar measure on G is finite and every coefficient of this representation is a bounded function). They are ruled by the Peter-Weyl theorem [22, 27]. Precisely, the Hilbert space $\mathrm{L}^2(G)$ admits the following orthogonal sum decomposition

$$\mathrm{L}^2(G) = \bigoplus_{U \in \check{G}} \mathrm{L}^2(G)_{[U]}\,, \tag{17}$$

where $\mathrm{L}^2(G)_{[U]}$ is a (closed) subspace of $\mathrm{L}^2(G)$, characterized by the following properties:

  1. $\mathrm{L}^2(G)_{[U]}$ depends only on the unitary equivalence class $[U]$ of U and it is an invariant subspace for the left regular representation R of G;
  2. for every orthonormal basis $\{\chi_n\}_{n=1}^{\delta(U)}$ in the Hilbert space of the representation $U \in \check{G}$, we have that

$$\mathrm{L}^2(G)_{[U]} = \bigoplus_{n=1}^{\delta(U)} \mathrm{Ran}\big(\mathfrak{W}_U^{\chi_n}\big) \tag{18}$$

  — hence: $\dim\big(\mathrm{L}^2(G)_{[U]}\big) = \delta(U)^2$; moreover, $\mathrm{Ran}\big(\mathfrak{W}_U^{\chi_n}\big)$ is an invariant subspace for the regular representation R and the restriction of R to $\mathrm{Ran}\big(\mathfrak{W}_U^{\chi_n}\big)$ is unitarily equivalent to U; therefore, 'U appears with multiplicity $\delta(U)$ in the left regular representation R', namely, R is unitarily equivalent to the representation

$$\bigoplus_{U \in \check{G}} \overbrace{U \oplus \cdots \oplus U}^{\delta(U)}; \tag{19}$$

  3. assuming that the Haar measure μ_G is normalized as usual for compact groups — i.e. $\mu_G(G) = 1$ — for any ϕ, ψ we have:

$$\delta(U) \int_G \langle \phi_1, U(g)\, \psi_1 \rangle \, \langle U(g)\, \psi_2, \phi_2 \rangle \; \mathrm{d}\mu_G(g) = \langle \phi_1, \phi_2 \rangle \, \langle \psi_2, \psi_1 \rangle\,; \tag{20}$$

  hence, the Duflo-Moore operator associated with the unitary representation U is of the form $d_U\, I$, where $d_U = \delta(U)^{-\frac{1}{2}}$.

- Let $\hat{q}$, $\hat{p}$ the standard position and momentum operators in $\mathrm{L}^2(\mathbb{R})$. Then, the map

$$U \colon \mathbb{R} \times \mathbb{R} \ni (q,p) \mapsto \exp(\mathrm{i}\,(p\,\hat{q} - q\hat{p})) \in \mathcal{U}(\mathrm{L}^2(\mathbb{R})) \tag{21}$$

  is a projective representation of the (additive) group $\mathbb{R} \times \mathbb{R}$. This representation is square integrable and, fixing $(2\pi)^{-1}\mathrm{d}q\mathrm{d}p$ as the Haar measure on $\mathbb{R} \times \mathbb{R}$, we have that $\hat{D}_U = I$; see [17]. Therefore, the Haar measure $(2\pi)^{-1}\mathrm{d}q\mathrm{d}p$ is normalized in agreement with U. If $\psi_0 \in \mathrm{L}^2(\mathbb{R})$ is the ground state of the quantum harmonic oscillator, then $\{U(q,p)\,\psi_0\}_{q,p\in\mathbb{R}}$ is the family of standard *coherent states* [28, 29].

For the reader's convenience, we conclude this section fixing some further notations and recalling a technical result that will be useful later on.

If $\hat{C}$ is a closable operator in $\mathcal{H}$, the symbol $\overline{\hat{C}}$ will indicate the closure of $\hat{C}$; a *core* for $\hat{C}$ is a linear span in $\mathcal{H}$, contained in the domain $\mathrm{Dom}(\hat{C})$, such that the closure of the restriction

of $\hat{C}$ to this linear span coincides with the closure of $\hat{C}$. For any pair of selfadjoint operators $\hat{A}, \hat{B}$ in $\mathcal{H}$, with a slight abuse of notation we will denote by $\hat{A} \otimes \hat{B}$ the *closure* of the ordinary tensor product of $\hat{A}$ by $\hat{B}$, closure which is a selfadjoint operator. Given a subspace $\mathcal{S}$ of $\mathcal{H}$, we will denote by $\mathcal{S}^{\perp}$ the orthogonal complement of $\mathcal{S}$ in $\mathcal{H}$. We will denote by $\mathcal{B}(\mathcal{H})$ the Banach space of bounded linear operators in the Hilbert space $\mathcal{H}$ and by $\|\cdot\|$ the associated norm. We recall that the Hilbert space of Hilbert-Schmidt operators $\mathcal{B}_2(\mathcal{H})$ in $\mathcal{H}$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$ [30]; the associated scalar product and norm will be denoted by $\langle\cdot,\cdot\rangle_{\mathcal{B}_2}$ and $\|\cdot\|_{\mathcal{B}_2}$, respectively. Another two-sided ideal in $\mathcal{B}(\mathcal{H})$ is the Banach space of trace class operators $\mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}_2(\mathcal{H})$. We will often use Dirac's notation for rank-one operators: $|\phi\rangle\langle\psi|\,\chi := \langle\psi,\chi\rangle\,\phi$, for any $\phi,\psi,\chi \in \mathcal{H}$.

Given a measure space (X,μ) the locution "for μ-almost all x in X" will be usually substituted by the symbol $\forall_\mu x \in X$. The following well known result will turn out to be very useful for our purposes in Sect. 5. Let the measure space (X,μ) be complete, and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathrm{L}^2(X,\mu;\mathbb{C})$ converging (in norm) to f. If there is a function $\tilde{f}\colon X \to \mathbb{C}$ such that $\lim_{n\to\infty} f_n(x) = \tilde{f}(x)$, $\forall_\mu x \in X$, then $\tilde{f}$ is μ-measurable and we have: $f = \tilde{f}$, the two functions being regarded as elements of $\mathrm{L}^2(X,\mu;\mathbb{C})$ (i.e. the two functions coincide μ-almost everywhere).[4]

3 Weyl-Wigner quantization-dequantization maps

As we have recalled in the previous section, with every square integrable representation of a l.c.s.c. group G one can associate an isometry — the (generalized) wavelet transform — mapping the Hilbert space of the representation into the space $\mathrm{L}^2(G)$. Beside this map, one can define another important isometry. This isometry maps the space of Hilbert-Schmidt operators — acting in the Hilbert space of the representation — into $\mathrm{L}^2(G)$. Since it transforms operators into functions, it is called the *Wigner (dequantization) map*. Its adjoint, which transforms functions into operators, is called the *Weyl (quantization) map*.

Indeed, we recall that — see [17, 18, 19] — given a square integrable projective representation $U\colon G \to \mathcal{U}(\mathcal{H})$ (with multiplier $\mathtt{m}$), with every Hilbert-Schmidt operator $\hat{A} \in \mathcal{B}_2(\mathcal{H})$ one can suitably associate a function

$$G \ni g \mapsto \big(\mathfrak{S}_U \hat{A}\big)(g) \in \mathbb{C} \tag{22}$$

contained in $\mathrm{L}^2(G) \equiv \mathrm{L}^2(G,\mu_G;\mathbb{C})$. Denoting by $\hat{D}_U$, as in Sect. 2, the Duflo-Moore operator associated with U (normalized according to a left Haar measure μ_G on G), consider the following *formal* definition: $\big(\mathfrak{S}_U \hat{A}\big)(g) := \mathsf{tr}\big(U(g)^* \hat{A}\, \hat{D}_U^{-1}\big)$. Since the operator $U(g)^* \hat{A}\, \hat{D}_U^{-1}$ (or, possibly, its closure) is not, generally speaking, a trace class operator, the given definition requires a rigorous interpretation. This can be achieved by suitably restricting the class of Hilbert Schmidt operators for which the definition makes sense, and then extending 'by density' the map obtained in such a way. To this aim, one can exploit the fact that the *finite rank operators* form a dense linear span $\mathsf{FR}(\mathcal{H})$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$.

Precisely, consider those rank one operators in $\mathcal{H}$ that are of the type

$$\widehat{\phi\psi} = |\phi\rangle\langle\psi|, \quad \phi \in \mathcal{H},\ \psi \in \mathrm{Dom}\big(\hat{D}_U^{-1}\big). \tag{23}$$

The linear span generated by the operators of this form, namely, the set

$$\mathsf{FR}^{\langle|}(\mathcal{H};U) := \big\{\hat{F} \in \mathsf{FR}(\mathcal{H}) :\ \mathrm{Ran}(\hat{F}^*) = \mathrm{Ker}(\hat{F})^{\perp} \subset \mathrm{Dom}\big(\hat{D}_U^{-1}\big)\big\}, \tag{24}$$

[4]This result is a consequence of the fact that the convergence with respect to the norm of $\mathrm{L}^2(X,\mu;\mathbb{C})$ implies the convergence in μ-measure.

is dense in $\mathsf{FR}(\mathcal{H})$ and, hence, in $\mathcal{B}_2(\mathcal{H})$:

$$\overline{\mathsf{FR}^{\langle|}(\mathcal{H};U)} = \mathcal{B}_2(\mathcal{H}). \tag{25}$$

Explicitly, the elements of $\mathsf{FR}^{\langle|}(\mathcal{H};U)$ are those operators in $\mathsf{FR}(\mathcal{H})$ that admit a *canonical decomposition* of the form

$$\hat{F} = \sum_{k=1}^{\mathsf{N}} |\phi_k\rangle\langle\psi_k|, \quad \mathsf{N} \in \mathbb{N}, \tag{26}$$

where $\{\phi_k\}_{k=1}^{\mathsf{N}}$, $\{\psi_k\}_{k=1}^{\mathsf{N}}$ are linearly independent systems in $\mathcal{H}$, with $\{\psi_k\}_{k=1}^{\mathsf{N}} \subset \mathrm{Dom}\big(\hat{D}_U^{-1}\big)$. Later on, it will also turn out to be useful the definition of the following dense linear span in $\mathcal{B}_2(\mathcal{H})$:

$$\mathsf{FR}^{|\rangle\langle|}(\mathcal{H};U) := \big\{\hat{F} \in \mathsf{FR}(\mathcal{H}):\ \mathrm{Ran}(\hat{F}), \mathrm{Ran}(\hat{F}^*) \subset \mathrm{Dom}\big(\hat{D}_U^{-1}\big)\big\}. \tag{27}$$

Observe now that, if we set

$$\big(\mathfrak{S}_U\, \widehat{\phi\psi}\big)(g) := \mathsf{tr}\big(U(g)^*|\phi\rangle\langle\hat{D}_U^{-1}\psi|\big) = \big\langle U(g)\, \hat{D}_U^{-1}\psi, \phi\big\rangle, \quad \forall \widehat{\phi\psi} \in \mathsf{FR}^{\langle|}(\mathcal{H};U), \tag{28}$$

then, by virtue of the orthogonality relations (8), for any $\widehat{\phi_1\psi_1} \equiv |\phi_1\rangle\langle\psi_1|, \widehat{\phi_2\psi_2} \in \mathsf{FR}^{\langle|}(\mathcal{H};U)$ we have:

$$\begin{aligned}\int_G \big(\mathfrak{S}_U\, \widehat{\phi_1\psi_1}\big)(g)^*\, \big(\mathfrak{S}_U\, \widehat{\phi_2\psi_2}\big)(g)\; \mathrm{d}\mu_G(g) \;&=\; \int_G \big\langle \phi_1, U(g)\, \hat{D}_U^{-1}\psi_1\big\rangle \big\langle U(g)\, \hat{D}_U^{-1}\psi_2, \phi_2\big\rangle\; \mathrm{d}\mu_G(g) \\ &= \langle\phi_1,\phi_2\rangle\, \langle\psi_2,\psi_1\rangle = \big\langle \widehat{\phi_1\psi_1}, \widehat{\phi_2\psi_2}\big\rangle_{\mathcal{B}_2}. \end{aligned} \tag{29}$$

Therefore, extending the map $\mathfrak{S}_U$ to all $\mathsf{FR}(\mathcal{H};U)$ by linearity, and then to the whole Hilbert space $\mathcal{B}_2(\mathcal{H})$ by continuity, we obtain an *isometry*

$$\mathfrak{S}_U \colon \mathcal{B}_2(\mathcal{H}) \to \mathrm{L}^2(G) \tag{30}$$

called the *(generalized) Wigner map*, or *Wigner transform*, generated by the square integrable representation U. We will denote by $\mathcal{R}_U$ the range of the isometry $\mathfrak{S}_U$. It is easy to check that $\mathcal{R}_U$ depends only on the unitary equivalence class of U. As the reader may prove, if the group G is unimodular (hence: $\hat{D}_U = d_U I$, with $d_U > 0$), then for every trace class operator $\hat{\rho} \in \mathcal{B}_1(\mathcal{H})$ — in particular, for every density operator in $\mathcal{H}$ — we have simply:

$$\big(\mathfrak{S}_U\, \hat{\rho}\big)(g) = d_U^{-1}\, \mathsf{tr}\big(U(g)^*\hat{\rho}\big). \tag{31}$$

Remark 2 Suppose that U is, in particular, a standard unitary representation, and let V be another square integrable unitary representation of G (acting in a Hilbert space $\mathcal{H}'$), unitarily *inequivalent* to U. Then, it is easy to show that

$$\big(\mathcal{R}_U \equiv \mathrm{Ran}(\mathfrak{S}_U)\big) \perp \mathrm{Ran}\big(\mathfrak{S}_V\big), \tag{32}$$

where $\mathfrak{S}_V$ is the Wigner map generated by V. Indeed, let $\mathfrak{W}_U^{\psi}$ and $\mathfrak{W}_V^{\eta}$ be the wavelet transforms generated by U and V, with analyzing vectors $\psi \in \mathcal{H}$ and $\eta \in \mathcal{H}'$, respectively. Relation (16) (with $\mathtt{m} \equiv 1$) implies that the bounded linear map $\mathfrak{W}_V^{\eta *}\, \mathfrak{W}_U^{\psi} \colon \mathcal{H} \to \mathcal{H}'$ intertwines the unitary representation U with V; hence, by Schur's lemma, it must be identically zero. Therefore, we have that

$$0 = \big\langle \mathfrak{W}_V^{\eta *}\, \mathfrak{W}_U^{\psi}\phi, \xi\big\rangle = \big\langle \mathfrak{W}_U^{\psi}\phi, \mathfrak{W}_V^{\eta}\xi\big\rangle, \quad \forall \phi \in \mathcal{H},\ \forall \xi \in \mathcal{H}'; \tag{33}$$

i.e. $\mathrm{Ran}\big(\mathfrak{W}_U^{\psi}\big) \perp \mathrm{Ran}\big(\mathfrak{W}_V^{\eta}\big)$. At this point, relation (32) follows observing that

$$\mathcal{R}_U = \overline{\mathrm{span}\big\{f \in \mathrm{Ran}\big(\mathfrak{W}_U^{\psi}\big):\ \psi \in \mathsf{A}(U),\ \psi \neq 0\big\}}, \tag{34}$$

and, of course, an analogous relation holds for the range of $\mathfrak{S}_V$. ■

Remark 3 Suppose that the group G is compact — hence, unimodular — and U is a (irreducible) *unitary* representation. Then, by relation (18), we have:

$$\mathrm{L}^2(G)_{[U]} = \bigoplus_{n=1}^{\delta(U)} \mathrm{Ran}\big(\mathfrak{W}_U^{\chi_n}\big) = \mathrm{span}\left\{\mathsf{c}^U_{\psi,\phi}\colon \psi,\phi\in\mathcal{H}\right\} = \mathcal{R}_U\,, \tag{35}$$

where the function $\mathsf{c}^U_{\psi,\phi}\in \mathrm{L}^2(G)$ is the coefficient defined by (5). Therefore, by relation (17), we conclude that

$$\mathrm{L}^2(G) = \bigoplus_{U\in\check{G}} \mathcal{R}_U\,, \tag{36}$$

where we recall that the symbol $\check{G}$ denotes a realization of the unitary dual of G. ■

We will now explore the 'intertwining properties' of the Wigner map $\mathfrak{S}_U$ with respect to the natural action of the group G in the Hilbert-Schmidt space $\mathcal{B}_2(\mathcal{H})$, and to the standard complex conjugation in $\mathcal{B}_2(\mathcal{H})$. To this aim, let us consider the map

$$U\vee U\colon G\rightarrow \mathcal{U}(\mathcal{B}_2(\mathcal{H})) \tag{37}$$

defined by

$$U\vee U(g)\,\hat{A} := U(g)\,\hat{A}\,U(g)^*, \quad \forall\, g\in G,\quad \hat{A}\in\mathcal{B}_2(\mathcal{H}). \tag{38}$$

The map $U\vee U$ is a (strongly continuous) *unitary* representation — even if, in general, the representation U has only been assumed to be *projective* — which can be regarded as the standard action of the 'symmetry group' G on the 'quantum-mechanical operators' ('observables' or 'states'). Next, let us consider the map

$$\mathcal{T}_{\mathtt{m}}\colon G\rightarrow \mathcal{U}(\mathrm{L}^2(G)) \tag{39}$$

defined by

$$\big(\mathcal{T}_{\mathtt{m}}(g)f\big)(g') := \Delta_G(g)^{\frac{1}{2}}\,\overleftrightarrow{\mathtt{m}}(g,g')\,f(g^{-1}g'g), \tag{40}$$

where the function $\overleftrightarrow{\mathtt{m}}: G\times G\rightarrow\mathbb{T}$ has the following expression:

$$\overleftrightarrow{\mathtt{m}}(g,g') := \mathtt{m}\,(g,g^{-1}g')^*\,\mathtt{m}\,(g^{-1}g',g). \tag{41}$$

As the reader may check by means of a direct calculation involving multipliers, the map $\mathcal{T}_{\mathtt{m}}$ is a unitary representation; the presence of the square root of the modular function Δ_G in formula (40) takes into account the right action of G on itself. Notice that, for $\mathtt{m}\equiv 1$, it coincides with the restriction to the 'diagonal subgroup' of the *two-sided regular representation* of the direct product group $G\times G$; see [22, 31].

Proposition 1 *The Wigner transform $\mathfrak{S}_U$ intertwines the representation $U\vee U$ with the representation $\mathcal{T}_{\mathtt{m}}$; namely,*

$$\mathfrak{S}_U\,U\vee U(g) = \mathcal{T}_{\mathtt{m}}(g)\,\mathfrak{S}_U\,, \quad \forall\, g\in G. \tag{42}$$

Therefore, $\mathcal{R}_U$ is an invariant subspace for the unitary representation $\mathcal{T}_{\mathtt{m}}$ and the representation $U\vee U$ is unitarily equivalent to a subrepresentation of $\mathcal{T}_{\mathtt{m}}$, i.e. to the restriction of $\mathcal{T}_{\mathtt{m}}$ to $\mathcal{R}_U$.

Proof: Let us first prove that $\mathfrak{S}_U\,U\vee U(g)\,\widehat{\phi\psi} = \mathcal{T}_{\mathtt{m}}(g)\,\mathfrak{S}_U\,\widehat{\phi\psi}$ for any rank-one operator $\widehat{\phi\psi}$ of the form

$$\widehat{\phi\psi}\equiv|\phi\rangle\langle\psi|, \quad \text{with } \phi\in\mathcal{H},\ \psi\in\mathrm{Dom}\big(\hat{D}_U^{-1}\big). \tag{43}$$

Observe that, for every $g \in G$, we have:

$$\big(\mathfrak{S}_U\, U \vee U(g)\, \widehat{\phi\psi}\big)(g') = \big(\mathfrak{S}_U\, |U(g)\,\phi\rangle\langle U(g)\,\psi|\big)(g') = \big\langle U(g')\,\hat{D}_U^{-1}\,U(g)\,\psi, U(g)\,\phi\big\rangle$$
$$= \big\langle U(g)^*\,U(g')\,\hat{D}_U^{-1}\,U(g)\,\psi, \phi\big\rangle. \quad (44)$$

At this point, we can exploit relation (11) and the standard properties of multipliers:

$$\begin{aligned}
\big(\mathfrak{S}_U\, U \vee U(g)\, \widehat{\phi\psi}\big)(g') &= \Delta_G(g)^{\frac{1}{2}}\,\big\langle U(g)^*\,U(g')\,U(g)\,\hat{D}_U^{-1}\,\psi, \phi\big\rangle \\
&= \Delta_G(g)^{\frac{1}{2}}\,\mathtt{m}\,(g, g^{-1})^*\,\big\langle U(g^{-1})\,U(g')\,U(g)\,\hat{D}_U^{-1}\,\psi, \phi\big\rangle \\
&= \Delta_G(g)^{\frac{1}{2}}\,\mathtt{m}\,(g, g^{-1})^*\,\mathtt{m}\,(g^{-1}, g')\,\mathtt{m}\,(g^{-1}g', g)\,\big\langle U(g^{-1}g'g)\,\hat{D}_U^{-1}\,\psi, \phi\big\rangle \\
&= \Delta_G(g)^{\frac{1}{2}}\,\mathtt{m}\,(g, g^{-1}g')^*\,\mathtt{m}\,(g^{-1}g', g)\,\big\langle U(g^{-1}g'g)\,\hat{D}_U^{-1}\,\psi, \phi\big\rangle \\
&= \big(\mathcal{T}_{\mathtt{m}}(g)\,\mathfrak{S}_U\,\widehat{\phi\psi}\big)(g'). \qquad (45)
\end{aligned}$$

This relation extends to the linear span generated by the rank-one operators of the form (43); i.e. to the dense linear span $\mathsf{FR}^{\langle|}(\mathcal{H}; U)$. Therefore, the bounded operators $\mathfrak{S}_U\, U \vee U(g)$ and $\mathcal{T}_{\mathtt{m}}(g)\,\mathfrak{S}_U$ coincide on a dense linear span in $\mathcal{B}_2(\mathcal{H})$; hence, they are equal. □

Remark 4 By a procedure analogous to the one adopted for proving relation (42) one can check that

$$\mathfrak{S}_U\big(U(g)\,\hat{A}\big) = R_{\mathtt{m}}(g)\big(\mathfrak{S}_U\hat{A}\big), \quad \forall \hat{A} \in \mathcal{B}_2(\mathcal{H}),\ \forall g \in G. \qquad (46)$$

This relation will be useful in Sect. 5. ■

Let us consider, now, the antilinear map $\mathsf{J}_{\mathtt{m}} : \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ defined by

$$\big(\mathsf{J}_{\mathtt{m}}\, f\big)(g) := \Delta_G(g)^{-\frac{1}{2}}\,\mathtt{m}\,(g, g^{-1})\, f(g^{-1})^*, \quad \forall f \in \mathrm{L}^2(G). \qquad (47)$$

We leave to the reader the easy task of verifying that the map $\mathsf{J}_{\mathtt{m}}$ is (well defined and) a *complex conjugation* in $\mathrm{L}^2(G)$: $\mathsf{J}_{\mathtt{m}} = \mathsf{J}_{\mathtt{m}}^*$ and $\mathsf{J}_{\mathtt{m}}^2 = I$ (i.e. $\mathsf{J}_{\mathtt{m}}$ is a selfadjoint antiunitary map).

Proposition 2 *The isometry $\mathfrak{S}_U$ intertwines the standard complex conjugation*

$$\mathfrak{J} : \mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \hat{A}^* \in \mathcal{B}_2(\mathcal{H}) \qquad (48)$$

in the Hilbert space $\mathcal{B}_2(\mathcal{H})$ with the complex conjugation $\mathsf{J}_{\mathtt{m}}$ in $\mathrm{L}^2(G)$; namely,

$$\mathfrak{S}_U\,\mathfrak{J} = \mathsf{J}_{\mathtt{m}}\,\mathfrak{S}_U. \qquad (49)$$

Therefore, $\mathcal{R}_U$ is an invariant subspace for the complex conjugation $\mathsf{J}_{\mathtt{m}}$.

Proof: We will first prove that $\mathfrak{S}_U\,\mathfrak{J}\,\widehat{\phi\psi} = \mathsf{J}_{\mathtt{m}}\,\mathfrak{S}_U\,\widehat{\phi\psi}$ for any rank-one operator $\widehat{\phi\psi}$ of the form

$$\widehat{\phi\psi} \equiv |\phi\rangle\langle\psi|, \quad \text{with } \phi, \psi \in \mathrm{Dom}\big(\hat{D}_U^{-1}\big). \qquad (50)$$

Observe that we have:

$$\big(\mathfrak{S}_U\,\mathfrak{J}\,\widehat{\phi\psi}\big)(g) = \big(\mathfrak{S}_U\,|\psi\rangle\langle\phi|\big)(g) = \big\langle U(g)\,\hat{D}_U^{-1}\,\phi, \psi\big\rangle = \big\langle \hat{D}_U^{-1}\,U(g)^*\,\psi, \phi\big\rangle^*$$
$$= \Delta_G(g)^{-\frac{1}{2}}\,\big\langle U(g)^*\,\hat{D}_U^{-1}\,\psi, \phi\big\rangle^*, \quad (51)$$

where for obtaining the last equality we have used relation (11). Then, taking into account that $U(g)^* = U(g)^{-1} = \mathtt{m}\,(g, g^{-1})\,U(g^{-1})$, we find:

$$\big(\mathfrak{S}_U\,\mathfrak{J}\,\widehat{\phi\psi}\big)(g) = \Delta_G(g)^{-\frac{1}{2}}\,\mathtt{m}\,(g, g^{-1})\,\langle U(g^{-1})\,\hat{D}_U^{-1}\,\psi, \phi\rangle^* = \big(\mathsf{J}_{\mathtt{m}}\,\mathfrak{S}_U\,\widehat{\phi\psi}\big)(g). \qquad (52)$$

Extending this relation to the linear span generated by the rank-one operators of the form (50) — i.e. to the dense linear span $\mathsf{FR}^{|\rangle\langle|}(\mathcal{H};U)$ in $\mathcal{B}_2(\mathcal{H})$ — and then, by continuity, to the whole $\mathcal{B}_2(\mathcal{H})$ one completes the proof. □

Since the generalized Wigner transform $\mathfrak{S}_U$ is an isometry, the adjoint map

$$\mathfrak{S}_U^*\colon \mathrm{L}^2(G) \to \mathcal{B}_2(\mathcal{H}) \tag{53}$$

is a partial isometry such that

$$\mathfrak{S}_U^*\,\mathfrak{S}_U = I, \quad \mathfrak{S}_U\,\mathfrak{S}_U^* = \mathrm{P}_{\mathcal{R}_U}, \tag{54}$$

where $\mathrm{P}_{\mathcal{R}_U}$ is the orthogonal projection onto the subspace $\mathcal{R}_U \equiv \mathrm{Ran}(\mathfrak{S}_U) = \mathrm{Ker}(\mathfrak{S}_U^*)$ of $\mathrm{L}^2(G)$. Thus, the partial isometry $\mathfrak{S}_U^*$ is the pseudo-inverse of $\mathfrak{S}_U$ and we will call it *(generalized) Weyl map* associated with the representation U.

Let us provide an expression of the Weyl map. As is well known, the weak integral

$$\hat{U}(\mathsf{f}) := \int_G \mathsf{f}(g)\, U(g)\; \mathrm{d}\mu_G(g), \quad \forall\, \mathsf{f} \in \mathrm{L}^1(G), \tag{55}$$

defines a bounded operator in $\mathcal{H}$ (here the square-integrability of U does not play any role). Then, one can easily prove the following result:

Proposition 3 *For every $\mathsf{f} \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$, the densely defined operator $\hat{U}(\mathsf{f})\,\hat{D}_U^{-1}$ extends to a Hilbert-Schmidt operator and*

$$\overline{\hat{U}(\mathsf{f})\,\hat{D}_U^{-1}} = \mathfrak{S}_U^*\,\mathsf{f}. \tag{56}$$

Therefore, for every function $f \in \mathrm{L}^2(G)$ — given a sequence $\{\mathsf{f}_n\}_{n\in\mathbb{N}}$ in $\mathrm{L}^2(G)$, contained in the dense linear span $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$, such that ${}_{\|\cdot\|_{\mathrm{L}^2}}\lim_{n\to\infty} \mathsf{f}_n = f$ — we have:

$$\mathfrak{S}_U^*\, f = {}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{n\to\infty} \mathfrak{S}_U^*\,\mathsf{f}_n = {}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{n\to\infty} \overline{\hat{U}(\mathsf{f}_n)\,\hat{D}_U^{-1}}. \tag{57}$$

In the case where the group G is unimodular, the following weak integral formula holds:

$$\mathfrak{S}_U^*\, f = d_U^{-1} \int_G f(g)\, U(g)\; \mathrm{d}\mu_G(g), \quad \forall\, f \in \mathrm{L}^2(G). \tag{58}$$

We will now prove a result that will be useful in Sect. 6.

Proposition 4 *Suppose that the Hilbert space $\mathcal{H}$ where the representation U acts is a space $\mathrm{L}^2(X) \equiv \mathrm{L}^2(X,\mu;\mathbb{C})$ of square integrable functions on a σ-finite measure space (X,μ). Then, for every $\mathsf{f} \in \mathrm{L}^1(G)$ and every $\phi \in \mathrm{L}^2(X)$, the function $G \ni g \mapsto \mathsf{f}(g)\,\big(U(g)\,\phi\big)(x) \in \mathbb{C}$ belongs to $\mathrm{L}^1(G)$ for μ-a.a. $x \in X$, and the following relation holds:*

$$\big(\hat{U}(\mathsf{f})\,\phi\big)(x) = \int_G \mathsf{f}(g)\,\big(U(g)\,\phi\big)(x)\; \mathrm{d}\mu_G(g), \quad \forall_\mu x \in X. \tag{59}$$

Therefore, for every $\mathsf{f} \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ and every $\varphi \in \mathrm{Dom}\big(\hat{D}_U^{-1}\big) \subset \mathrm{L}^2(X)$, we have:

$$\Big(\big(\mathfrak{S}_U^*\,\mathsf{f}\big)\,\varphi\Big)(x) = \int_G \mathsf{f}(g)\,\big(U(g)\,\hat{D}_U^{-1}\,\varphi\big)(x)\; \mathrm{d}\mu_G(g), \quad \forall_\mu x \in X. \tag{60}$$

Proof: By definition of the operator $\hat{U}(\mathsf{f})$ (which involves a weak integral), we have that, for every $\mathsf{f}\in \mathrm{L}^1(G)$ and any $\phi,\psi\in\mathrm{L}^2(X)$,

$$\langle\psi,\hat{U}(\mathsf{f})\,\phi\rangle = \langle\psi,\int_G \mathsf{f}(g)\,U(g)\;\mathrm{d}\mu_G(g)\;\phi\rangle \;=\; \int_G \mathsf{f}(g)\,\langle\psi,U(g)\,\phi\rangle\;\mathrm{d}\mu_G(g)$$
$$= \int_G \mathrm{d}\mu_G(g)\int_X \mathrm{d}\mu(x)\;\mathsf{f}(g)\,\psi(x)^*\big(U(g)\,\phi\big)(x)\,. \quad (61)$$

By the arbitrariness of $\psi\in\mathrm{L}^2(X)$, relation (59) can be proved showing that

$$\langle\psi,\int_G \mathsf{f}(g)\,U(g)\;\mathrm{d}\mu_G(g)\;\phi\rangle = \int_X \mathrm{d}\mu(x)\;\psi(x)^*\int_G \mathrm{d}\mu_G(g)\;\mathsf{f}(g)\,\big(U(g)\,\phi\big)(x)\,, \quad (62)$$

namely, that the iterated integrals in the last member of (61) can be permuted. In fact, since (X,μ) (like (G,μ_G)) is a σ-finite measure space, we can use Tonelli's theorem; by this theorem, the Schwarz inequality and the fact that $\|U(g)\,\phi\|=\|\phi\|$, we have:

$$\int_{G\times X}\big|\psi(x)^*\,\mathsf{f}(g)\,\big(U(g)\,\phi\big)(x)\big|\;\mathrm{d}\mu_G\!\otimes\mu\,(g,x) \;=\; \int_G \mathrm{d}\mu_G(g)\;|\mathsf{f}(g)|\int_X \mathrm{d}\mu(x)\;|\psi(x)|\,\big|\big(U(g)\,\phi\big)(x)\big|$$
$$\le \|\mathsf{f}\|_{\mathrm{L}^1}\,\|\psi\|\,\|\phi\|\,. \quad (63)$$

Therefore, the function $G\times X\ni(g,x)\mapsto\psi(x)^*\,\mathsf{f}(g)\,\big(U(g)\,\phi\big)(x)$ belongs to $\mathrm{L}^1(G\times X,\mu_G\otimes\mu;\mathbb{C})$, for any $\phi,\psi\in\mathrm{L}^2(X)$. It follows by Fubini's theorem that the function $G\ni g\mapsto\mathsf{f}(g)\,\big(U(g)\,\phi\big)(x)$ belongs to $\mathrm{L}^1(G)$, $\forall_\mu x\in X$, and the iterated integrals in the last member of (61) can indeed be permuted. $\square$

4 Star products from quantization-dequantization maps

In this section, we will show that the quantization-dequantization maps previously introduced induce, in a natural way, a 'star product of functions' enjoying remarkable properties. Let U be a square integrable projective representation of the l.c.s.c. group G in the Hilbert space $\mathcal{H}$, and let $\mathfrak{S}_U\colon\mathcal{B}_2(\mathcal{H})\to\mathrm{L}^2(G)$ be the associated Wigner map. Consider the following bilinear map from $\mathrm{L}^2(G)\times\mathrm{L}^2(G)$ into $\mathrm{L}^2(G)$:

$$(\cdot)\overset{U}{\star}(\cdot)\colon\mathrm{L}^2(G)\times\mathrm{L}^2(G)\ni(f_1,f_2)\mapsto\mathfrak{S}_U\Big(\big(\mathfrak{S}_U^*\,f_1\big)\big(\mathfrak{S}_U^*\,f_2\big)\Big)\in\mathrm{L}^2(G); \quad (64)$$

i.e. $f_1\overset{U}{\star}f_2$ is the function obtained dequantizing the product (composition) of the two operators which are the 'quantized versions' of the functions f_1, f_2. We will call the bilinear map (64) the *star product associated with the representation* U.

Before considering the properties of the star product associated with U, it is worth fixing some terminology about algebras. By a *Banach algebra* we mean an associative algebra $\mathcal{A}$ which is a Banach space (with norm $\|\cdot\|_{\mathcal{A}}$) such that

$$\|a\,b\|_{\mathcal{A}}\le\|a\|_{\mathcal{A}}\,\|b\|_{\mathcal{A}}\,,\quad\forall a,b\in\mathcal{A}. \quad (65)$$

Given Banach algebras $\mathcal{A}$ and $\mathcal{A}'$, we will say that a linear map $\mathfrak{E}\colon\mathcal{A}\to\mathcal{A}'$ is an (isometric) *isomorphism of Banach algebras* if it is a surjective isometry such that $\mathfrak{E}(a\,b)=\mathfrak{E}(a)\,\mathfrak{E}(b)$, for all $a,b\in\mathcal{A}$. A Banach algebra $\mathcal{A}$ — endowed with an involution[5] $(a\mapsto a^*)$ — such that

$$\|a\|_{\mathcal{A}}=\|a^*\|_{\mathcal{A}}\,,\quad\forall a\in\mathcal{A}, \quad (66)$$

[5] Let V be a vector space, and $(\cdot,\cdot)\colon\mathsf{V}\times\mathsf{V}\to\mathsf{V}$ a bilinear operation in V. We recall that an an *involution* in V, with respect to the bilinear operation $(\cdot,\cdot)$, is an antilinear map $\mathsf{V}\ni a\mapsto a^*\in\mathsf{V}$ satisfying: $(a^*)^*=a$ and $(a,b)^*=(b^*,a^*)$, $\forall a,b\in\mathsf{V}$.

will be called a *Banach $*$-algebra* (Banach star-algebra; of course, the 'star' in $*$-algebra, which refers to an involution, should not generate confusion with the 'star' product).

A Banach $*$-algebra $\mathcal{A}$ is said to be a H*-*algebra* [32, 33] if, in addition, it is a (separable complex) Hilbert space (with $\|a\|_{\mathcal{A}} = \sqrt{\langle a, a\rangle_{\mathcal{A}}}$) satisfying:

$$\langle a\, b, c\rangle_{\mathcal{A}} = \langle b, a^* c\rangle_{\mathcal{A}} \quad \text{and} \quad \langle a\, b, c\rangle_{\mathcal{A}} = \langle a, c\, b^*\rangle_{\mathcal{A}}, \quad \forall\, a, b, c \in \mathcal{A}; \tag{67}$$

Clearly, condition (66) now means that the involution $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ is a complex conjugation (an idempotent antiunitary operator).

Remark 5 The definition of a H*-algebra given above may seem to be stricter than the usual definition. In fact, one usually defines a H*-algebra as a Banach algebra $\mathcal{A}$ which is a Hilbert space and satisfies the following condition: for each $a \in \mathcal{A}$, there is an element $a^\diamond \in \mathcal{A}$ (which need not be unique) — an *adjoint* of a — such that

$$\langle a\, b, c\rangle_{\mathcal{A}} = \langle b, a^\diamond c\rangle_{\mathcal{A}} \quad \text{and} \quad \langle a\, b, c\rangle_{\mathcal{A}} = \langle a, c\, b^\diamond\rangle_{\mathcal{A}}, \quad \forall\, b, c \in \mathcal{A}. \tag{68}$$

Let us show that the two definitions are equivalent; i.e. that the *usual definition* implies the *strict definition*. We will use some terminology and results from [32, 33]. Let $\mathcal{A}$ be a H*-algebra, according to the usual definition. For every element x of a $\mathcal{A}$, the two relations $x\mathcal{A} = \{0\}$ and $\mathcal{A}x = \{0\}$ turn out to be equivalent. The *annihilator ideal* of $\mathcal{A}$ is the set defined by

$$\mathcal{A}_0 := \{x \in \mathcal{A}\colon\ x\mathcal{A} = \{0\}\} = \{x \in \mathcal{A}\colon\ \mathcal{A}x = \{0\}\}. \tag{69}$$

The annihilator ideal is a selfadjoint[6] closed two-sided ideal in $\mathcal{A}$; the set of all the adjoints of any $x \in \mathcal{A}_0$ is $\mathcal{A}_0$ itself. A H*-algebra $\mathcal{A}$ is said to be *proper* (or *semisimple*) if it satisfies the following two equivalent conditions:

$$(x \in \mathcal{A},\ \ x\mathcal{A} = \{0\} \ \ \Rightarrow\ \ x = 0) \quad \text{and} \quad (x \in \mathcal{A},\ \ \mathcal{A}x = \{0\} \ \ \Rightarrow\ \ x = 0)\,; \tag{70}$$

namely, if $\mathcal{A}_0 = \{0\}$. Every element y of a proper H*-algebra $\mathcal{A}$ has a *unique* adjoint — which we denote by $y^\bullet$ — and $\|y\|_{\mathcal{A}} = \|y^\bullet\|_{\mathcal{A}}$; moreover, the map $\mathcal{A} \ni y \mapsto y^\bullet \in \mathcal{A}$ is an involution. A H*-algebra $\mathcal{A}$ admits an orthogonal sum decomposition of the following type:

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1, \tag{71}$$

where $\mathcal{A}_0$ is the annihilator ideal of $\mathcal{A}$, and $\mathcal{A}_1$ is a closed two sided ideal which (endowed with the restriction of the algebra operation of $\mathcal{A}$) is a proper H*-algebra. We will call $\mathcal{A}_1$ the *canonical ideal* of $\mathcal{A}$, and we will denote by $\mathrm{P}_{\mathcal{A}_1}$ the orthogonal projection onto $\mathcal{A}_1$. The canonical ideal is characterized by the relation

$$a\, b = \big(\mathrm{P}_{\mathcal{A}_1} a\big)\big(\mathrm{P}_{\mathcal{A}_1} b\big), \quad \forall\, a, b \in \mathcal{A}, \tag{72}$$

in the following sense. Suppose that $\tilde{\mathcal{A}} \subset \mathcal{A}$ is a closed two-sided ideal, which is a proper H*-algebra such that $a\, b = \big(\mathrm{P}_{\tilde{\mathcal{A}}}\, a\big)\big(\mathrm{P}_{\tilde{\mathcal{A}}}\, b\big)$, $\forall\, a, b \in \mathcal{A}$. Then, it is easy to show that $\tilde{\mathcal{A}} = \mathcal{A}_1$.[7] Let $J_0\colon \mathcal{A}_0 \to \mathcal{A}_0$ be an arbitrary complex conjugation in the annihilator ideal, and $J_1\colon \mathcal{A}_1 \to \mathcal{A}_1$ the involution defined by $J_1\, y = y^\bullet$, for any $y \in \mathcal{A}_1$ — where $y^\bullet$ is the unique adjoint of y

[6] A subset $\mathcal{E}$ of $\mathcal{A}$ is said to be *selfadjoint* if the set $\mathcal{E}^\diamond$ consisting of all the adjoints of the elements of $\mathcal{E}$ coincides with the set $\mathcal{E}$ itself.

[7] Indeed, for each $a \in \tilde{\mathcal{A}}^\perp$, it is clear that $a\, b = 0$, $\forall\, b \in \mathcal{A}$. Hence, $\tilde{\mathcal{A}}^\perp \subset \mathcal{A}_0$ and $\tilde{\mathcal{A}} \supset \mathcal{A}_1$. Now, let c be a vector in $\tilde{\mathcal{A}}$. Then, $c = c_0 + c_1$, for some $c_0 \in \mathcal{A}_0$ and $c_1 \in \mathcal{A}_1$, and, since $\tilde{\mathcal{A}} \supset \mathcal{A}_1$, $c_0 = c - c_1 \in \tilde{\mathcal{A}}$. Therefore, $c_0 = 0$ as $\tilde{\mathcal{A}}$ is a proper H*-algebra. It follows that $\tilde{\mathcal{A}} = \mathcal{A}_1$.

belonging to $\mathcal{A}_1$[8] — which is a complex conjugation since y and $y^\bullet$ satisfy: $\|y\|_{\mathcal{A}} = \|y^\bullet\|_{\mathcal{A}}$. We can now define an antilinear map $a \mapsto a^*$ in $\mathcal{A}$ by setting:

$$a^* = (J_0 \oplus J_1)\, a\,, \quad \forall a \in \mathcal{A}. \tag{73}$$

It is clear that this map is an involution that verifies both conditions (66) and (67). It is easy to check that any involution $a \mapsto a^*$ in $\mathcal{A}$ satisfying (66) and (67) must be of the form (73). ■

A linear map $\mathfrak{E}\colon \mathcal{A} \to \mathcal{A}'$ — where $\mathcal{A}$, $\mathcal{A}'$ are H*-algebras — is said to be an *isomorphism of* H*-*algebras* if it is a unitary operator such that

$$\mathfrak{E}(a\,b) = \mathfrak{E}(a)\,\mathfrak{E}(b) \quad \text{and} \quad \mathfrak{E}(a^*) = \mathfrak{E}(a)^*, \quad \forall a, b \in \mathcal{A}. \tag{74}$$

As is well known, the Hilbert space $\mathcal{B}_2(\mathcal{H})$ is a proper H*-algebra with respect to the ordinary composition of operators (algebra operation) and to the standard complex conjugation $\mathfrak{J}$ (involution), see (48).

The star product defined above is characterized by the following result:

Proposition 5 *The bilinear map* $(\cdot) \overset{U}{\star} (\cdot)\colon \mathrm{L}^2(G) \times \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ *associated with the square integrable projective representation* U *enjoys the following properties:*

1. *the vector space* $\mathrm{L}^2(G)$*, endowed with the operation* $(\cdot) \overset{U}{\star} (\cdot)$*, is an associative algebra;*
2. *the antilinear map* $\mathsf{J}_{\mathtt{m}}$ *is an involution in the vector space* $\mathrm{L}^2(G)$ *with respect to the bilinear operation* $(\cdot) \overset{U}{\star} (\cdot)$*, i.e.*

$$\mathsf{J}_{\mathtt{m}}\left(\mathsf{J}_{\mathtt{m}} f\right) = f \quad \textit{and} \quad \mathsf{J}_{\mathtt{m}}\Big(f_1 \overset{U}{\star} f_2\Big) = \big(\mathsf{J}_{\mathtt{m}} f_2\big) \overset{U}{\star} \big(\mathsf{J}_{\mathtt{m}} f_1\big), \quad \forall f, f_1, f_2 \in \mathrm{L}^2(G); \tag{75}$$

3. $\mathrm{L}^2(G)$ *— regarded as a Banach space with respect to the norm* $\|\cdot\|_{\mathrm{L}^2}$*, and endowed with the the star product associated with* U *and with the involution* $\mathsf{J}_{\mathtt{m}}$ *— is a Banach ∗-algebra; in particular, it satisfies the relation*

$$\Big\| f_1 \overset{U}{\star} f_2 \Big\|_{\mathrm{L}^2} \le \|f_1\|_{\mathrm{L}^2} \|f_2\|_{\mathrm{L}^2}, \quad \forall f_1, f_2 \in \mathrm{L}^2(G); \tag{76}$$

4. $\mathcal{A}_U \equiv \big(\mathrm{L}^2(G), (\cdot) \overset{U}{\star} (\cdot), \mathsf{J}_{\mathtt{m}}\big)$ *is a* H*-*algebra; indeed, for all* $f_1, f_2, f_3 \in \mathrm{L}^2(G)$*,*

$$\langle f_1 \overset{U}{\star} f_2, f_3 \rangle_{\mathrm{L}^2} = \langle f_2, (\mathsf{J}_{\mathtt{m}} f_1) \overset{U}{\star} f_3 \rangle_{\mathrm{L}^2} \quad \textit{and} \quad \langle f_1 \overset{U}{\star} f_2, f_3 \rangle_{\mathrm{L}^2} = \langle f_1, f_3 \overset{U}{\star} (\mathsf{J}_{\mathtt{m}} f_2) \rangle_{\mathrm{L}^2}; \tag{77}$$

5. *for any* $f_1, f_2 \in \mathrm{L}^2(G)$*, we have that*

$$f_1 \overset{U}{\star} f_2 \in \mathcal{R}_U; \tag{78}$$

therefore, the (closed) subspace $\mathcal{R}_U \equiv \mathrm{Ran}\big(\mathfrak{S}_U\big)$ *of* $\mathrm{L}^2(G)$ *is a closed two-sided ideal in* $\mathcal{A}_U$ *and — endowed with the restrictions of the star product associated with* U *and of the involution* $\mathsf{J}_{\mathtt{m}}$ ($\mathcal{R}_U$ is an invariant subspace for $\mathsf{J}_{\mathtt{m}}$, see Proposition 2) *— is a* H*-*algebra;*

[8] It is clear that a generic adjoint of $y \in \mathcal{A}_1$ is of the form $y^\diamond = x + y^\bullet$, where x is an arbitrary element of $\mathcal{A}_0$.

6. *the* H^**-algebra* $\mathcal{R}_U$ *is proper and, for any* $f_1, f_2 \in \mathrm{L}^2(G)$*, we have that*

$$f_1 \overset{U}{\star} f_2 = \big(\mathrm{P}_{\mathcal{R}_U} f_1\big) \overset{U}{\star} \big(\mathrm{P}_{\mathcal{R}_U} f_2\big); \tag{79}$$

hence, $\mathcal{R}_U$ *and its orthogonal complement* $\mathcal{R}_U^\perp$ *are, respectively, the canonical ideal and the annihilator ideal of* $\mathcal{A}_U$*, and the* H^**-algebra* $\mathcal{A}_U$ *is proper if and only if* $\mathcal{R}_U = \mathrm{L}^2(G)$*;*

7. *the unitary operator*

$$\mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \mathfrak{S}_U \hat{A} \in \mathcal{R}_U \tag{80}$$

is an isomorphism of (proper) H^**-algebras;*

8. *the canonical ideal* $\mathcal{R}_U$ *is an invariant subspace for the representation* $\mathcal{T}_{\mathtt{m}}$ — (see (40)) — *and the star product associated with* U *is equivariant with respect to this representation, i.e.*

$$\mathcal{T}_{\mathtt{m}}(g)\Big(f_1 \overset{U}{\star} f_2\Big) = \big(\mathcal{T}_{\mathtt{m}}(g) f_1\big) \overset{U}{\star} \big(\mathcal{T}_{\mathtt{m}}(g) f_2\big), \quad \forall f_1, f_2 \in \mathrm{L}^2(G),\ \forall g \in G. \tag{81}$$

Proof: Since the star product $(\cdot) \overset{U}{\star} (\cdot)$ is by construction a bilinear map, the vector space $\mathrm{L}^2(G)$ — endowed with this operation — is an algebra. Let us prove that this algebra is associative. Indeed, observe that — using definition (64), and by virtue of the relation $\mathfrak{S}_U^* \mathfrak{S}_U = I$ and of the associativity of the H^*-algebra $\mathcal{B}_2(\mathcal{H})$ — for any $f_1, f_2, f_3 \in \mathrm{L}^2(G)$ we have:

$$\begin{aligned}\Big(f_1 \overset{U}{\star} f_2\Big) \overset{U}{\star} f_3 = \mathfrak{S}_U\Big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\Big) \overset{U}{\star} f_3 &= \mathfrak{S}_U\Big(\Big(\mathfrak{S}_U^* \mathfrak{S}_U\Big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\Big)\Big)\big(\mathfrak{S}_U^* f_3\big)\Big) \\ &= \mathfrak{S}_U\big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\big(\mathfrak{S}_U^* f_3\big)\big).\end{aligned} \tag{82}$$

From this relation, using again the fact that $\mathfrak{S}_U^* \mathfrak{S}_U = I$ and definition (64), we get

$$\begin{aligned}\Big(f_1 \overset{U}{\star} f_2\Big) \overset{U}{\star} f_3 &= \mathfrak{S}_U\Big(\big(\mathfrak{S}_U^* f_1\big)\Big(\mathfrak{S}_U^* \mathfrak{S}_U\Big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\Big)\Big)\Big) \\ &= \mathfrak{S}_U\Big(\big(\mathfrak{S}_U^* f_1\big)\Big(\mathfrak{S}_U^*\Big(f_2 \overset{U}{\star} f_3\Big)\Big)\Big) \\ &= f_1 \overset{U}{\star} \Big(f_2 \overset{U}{\star} f_3\Big), \quad \forall f_1, f_2, f_3 \in \mathrm{L}^2(G).\end{aligned} \tag{83}$$

Let us now prove that the antilinear map $\mathsf{J}_{\mathtt{m}}$ is an involution with respect to the bilinear operation $(\cdot) \overset{U}{\star} (\cdot)$. The first of relations (75) is certainly satisfied (since $\mathsf{J}_{\mathtt{m}}$ is a complex conjugation in $\mathrm{L}^2(G)$). In order to prove the second of relations (75), observe that the intertwining relation (49) implies that

$$\mathfrak{J}\, \mathfrak{S}_U^* = \mathfrak{J}^* \mathfrak{S}_U^* = \mathfrak{S}_U^* \mathsf{J}_{\mathtt{m}}^* = \mathfrak{S}_U^* \mathsf{J}_{\mathtt{m}}\,. \tag{84}$$

Then — using (49) and (84), and the fact that $\mathfrak{J}$ is an involution in the algebra $\mathcal{B}_2(\mathcal{H})$ — we can argue as follows:

$$\begin{aligned}\mathsf{J}_{\mathtt{m}}\Big(f_1 \overset{U}{\star} f_2\Big) = \mathsf{J}_{\mathtt{m}} \mathfrak{S}_U\Big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\Big) &= \mathfrak{S}_U \mathfrak{J}\Big(\big(\mathfrak{S}_U^* f_1\big)\big(\mathfrak{S}_U^* f_2\big)\Big) \\ &= \mathfrak{S}_U\Big(\big(\mathfrak{J}\, \mathfrak{S}_U^* f_2\big)\big(\mathfrak{J}\, \mathfrak{S}_U^* f_1\big)\Big) \\ &= \mathfrak{S}_U\Big(\big(\mathfrak{S}_U^* \mathsf{J}_{\mathtt{m}} f_2\big)\big(\mathfrak{S}_U^* \mathsf{J}_{\mathtt{m}} f_1\big)\Big) \\ &= \big(\mathsf{J}_{\mathtt{m}} f_2\big) \overset{U}{\star} \big(\mathsf{J}_{\mathtt{m}} f_1\big), \quad \forall f_1, f_2 \in \mathrm{L}^2(G).\end{aligned} \tag{85}$$

At this point, in order to show that $\mathcal{A}_U \equiv \big(\mathrm{L}^2(G), (\cdot) \overset{U}{\star} (\cdot), \mathsf{J}_{\mathtt{m}}\big)$ is a Banach $*$-algebra, it remains to observe that

$$\| \mathsf{J}_{\mathtt{m}} f \|_{\mathrm{L}^2} = \| f \|_{\mathrm{L}^2}, \quad \forall f \in \mathrm{L}^2(G), \tag{86}$$

($\mathsf{J}_{\mathtt{m}}$ is a antiunitary operator), and to prove relation (76). In fact, we have:

$$\begin{aligned} \Big\| f_1 \overset{U}{\star} f_2 \Big\|_{\mathrm{L}^2} = \Big\| \mathfrak{S}_U \Big(\big(\mathfrak{S}_U^* f_1\big) \big(\mathfrak{S}_U^* f_2\big) \Big) \Big\|_{\mathrm{L}^2} &= \big\| \big(\mathfrak{S}_U^* f_1\big) \big(\mathfrak{S}_U^* f_2\big) \big\|_{\mathcal{B}_2} \\ &\le \| \mathfrak{S}_U^* f_1 \|_{\mathcal{B}_2} \, \| \mathfrak{S}_U^* f_2 \|_{\mathcal{B}_2} \\ &\le \| f_1 \|_{\mathrm{L}^2} \, \| f_2 \|_{\mathrm{L}^2}, \quad \forall f_1, f_2 \in \mathrm{L}^2(G), \end{aligned} \tag{87}$$

where the last inequality is a consequence of the fact that $\mathfrak{S}_U^*$ is a partial isometry.

We will now prove that $\mathcal{A}_U$ is a H*-algebra. To this aim, let us show that the first of relations (77) holds true. In fact, for all $f_1, f_2, f_3 \in \mathrm{L}^2(G)$, we have that

$$\begin{aligned} \langle f_1 \overset{U}{\star} f_2, f_3 \rangle_{\mathrm{L}^2} = \big\langle \big(\mathfrak{S}_U^* f_1\big) \big(\mathfrak{S}_U^* f_2\big), \mathfrak{S}_U^* f_3 \big\rangle_{\mathcal{B}_2} &= \big\langle \mathfrak{S}_U^* f_2, \big(\mathfrak{S}_U^* f_1\big)^* \mathfrak{S}_U^* f_3 \big\rangle_{\mathcal{B}_2} \\ &= \big\langle f_2, \mathfrak{S}_U \big(\big(\mathfrak{J}\, \mathfrak{S}_U^* f_1\big) \mathfrak{S}_U^* f_3\big) \big\rangle_{\mathrm{L}^2} \\ &= \big\langle f_2, \mathfrak{S}_U \big(\big(\mathfrak{S}_U^* \mathsf{J}_{\mathtt{m}} f_1\big) \mathfrak{S}_U^* f_3\big) \big\rangle_{\mathrm{L}^2} \\ &= \langle f_2, (\mathsf{J}_{\mathtt{m}} f_1) \overset{U}{\star} f_3 \rangle_{\mathrm{L}^2}. \end{aligned} \tag{88}$$

The proof of the second of relations (77) is analogous; we leave the details to the reader.

Properties (78) and (79), and the fact that the map (80) is an isomorphism of H*-algebras can be immediately verified by the definition of the star product associated with U. Moreover, $\mathcal{R}_U$ is a *proper* H*-algebra, being isomorphic to $\mathcal{B}_2(\mathcal{H})$.

Eventually, we can prove the equivariance relation (81) by a procedure similar to (85). In fact, exploiting the intertwining relation (42), for every $f_1, f_2 \in \mathrm{L}^2(G)$, we have:

$$\begin{aligned} \mathcal{T}_{\mathtt{m}}(g) \Big(f_1 \overset{U}{\star} f_2 \Big) = \mathfrak{S}_U \, U \vee U(g) \Big(\big(\mathfrak{S}_U^* f_1\big) \big(\mathfrak{S}_U^* f_2\big) \Big) &= \mathfrak{S}_U \Big(\big(U \vee U(g) \, \mathfrak{S}_U^* f_2 \big) \big(U \vee U(g) \, \mathfrak{S}_U^* f_1 \big) \Big) \\ &= \mathfrak{S}_U \Big(\big(\mathfrak{S}_U^* \, \mathcal{T}_{\mathtt{m}}(g) \, f_1\big) \big(\mathfrak{S}_U^* \, \mathcal{T}_{\mathtt{m}}(g) \, f_2\big) \Big) \\ &= \big(\mathcal{T}_{\mathtt{m}}(g) \, f_1 \big) \overset{U}{\star} \big(\mathcal{T}_{\mathtt{m}}(g) \, f_2 \big). \end{aligned} \tag{89}$$

The proof is complete. □

It is interesting to note that the definition of the star product (64) can be suitably generalized. In fact, since $\mathcal{B}_2(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$, with every bounded operator $\hat{K} \in \mathcal{B}(\mathcal{H})$ is associated a bilinear map $(\cdot) \underset{\hat{K}}{\circ} (\cdot) \colon \mathcal{B}_2(\mathcal{H}) \times \mathcal{B}_2(\mathcal{H}) \to \mathcal{B}_2(\mathcal{H})$ — the $\hat{K}$*-product*[9] in $\mathcal{B}_2(\mathcal{H})$ — defined by

$$\hat{A} \underset{\hat{K}}{\circ} \hat{B} := \hat{A} \hat{K} \hat{B}, \quad \forall \hat{A}, \hat{B} \in \mathcal{B}_2(\mathcal{H}). \tag{90}$$

Observe that $\mathcal{B}_2(\mathcal{H})$, endowed with the operation $(\cdot) \underset{\hat{K}}{\circ} (\cdot)$, is an associative algebra, and, if $\hat{K}$ is selfadjoint, then $\mathfrak{J}$ is an involution in $\mathcal{B}_2(\mathcal{H})$ with respect to this operation. Moreover, since

$$\| \hat{A} \hat{K} \hat{B} \|_{\mathcal{B}_2} \le \| \hat{A} \|_{\mathcal{B}_2} \, \| \hat{K} \hat{B} \|_{\mathcal{B}_2} \le \| \hat{K} \| \, \| \hat{A} \|_{\mathcal{B}_2} \, \| \hat{B} \|_{\mathcal{B}_2}, \tag{91}$$

it is clear that, if $\|\hat{K}\| \le 1$, then $\big(\mathcal{B}_2(\mathcal{H}), (\cdot) \underset{\hat{K}}{\circ} (\cdot)\big)$ is a Banach algebra; if, furthermore, $\hat{K}$ is selfadjoint, then $\big(\mathcal{B}_2(\mathcal{H}), (\cdot) \underset{\hat{K}}{\circ} (\cdot), \mathfrak{J}\big)$ is a Banach $*$-algebra. The operation (90) allows us to introduce the following bilinear map from $\mathrm{L}^2(G) \times \mathrm{L}^2(G)$ into $\mathrm{L}^2(G)$:

$$(\cdot) \underset{\hat{K}}{\overset{U}{\star}} (\cdot) \colon \mathrm{L}^2(G) \times \mathrm{L}^2(G) \ni (f_1, f_2) \mapsto \mathfrak{S}_U \Big(\big(\mathfrak{S}_U^* f_1\big) \underset{\hat{K}}{\circ} \big(\mathfrak{S}_U^* f_2\big) \Big) \in \mathrm{L}^2(G). \tag{92}$$

[9] This notion has been considered for 'generic operators' in refs. [13, 14].

We will call the operation $(\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot)$ *$\hat{K}$-deformed star product associated with U*. Obviously, the $\hat{K}$-deformed star product coincides with the star product defined by (64) in the case where $\hat{K} = I$. By a procedure analogous to the one adopted in the proof of Proposition 5, one can derive the main properties of the $\hat{K}$-deformed star product:

Proposition 6 *For every bounded operator $\hat{K} \in \mathcal{B}(\mathcal{H})$, the bilinear map $(\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot)$ enjoys the following properties:*

1. *the vector space $\mathrm{L}^2(G)$, endowed with the operation $(\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot)$, is an associative algebra;*
2. *in the case where the operator $\hat{K}$ is selfadjoint, the antilinear map $\mathsf{J}_{\mathtt{m}}$ is an involution in the vector space $\mathrm{L}^2(G)$ with respect to the bilinear operation $(\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot)$, i.e.*

$$\mathsf{J}_{\mathtt{m}}\left(\mathsf{J}_{\mathtt{m}} f\right) = f \quad \text{and} \quad \mathsf{J}_{\mathtt{m}}\left(f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2\right) = \left(\mathsf{J}_{\mathtt{m}} f_2\right) \overset{U}{\underset{\hat{K}}{\star}} \left(\mathsf{J}_{\mathtt{m}} f_1\right), \quad \forall f, f_1, f_2 \in \mathrm{L}^2(G); \tag{93}$$

3. *if $\|\hat{K}\| \leq 1$, then $\mathrm{L}^2(G)$ — regarded as a Banach space with respect to the norm $\|\cdot\|_{\mathrm{L}^2}$, and endowed with the $\hat{K}$-deformed star product associated with U — is a Banach algebra; in particular, it satisfies the relation*

$$\left\| f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 \right\|_{\mathrm{L}^2} \leq \|f_1\|_{\mathrm{L}^2} \|f_2\|_{\mathrm{L}^2}, \quad \forall f_1, f_2 \in \mathrm{L}^2(G); \tag{94}$$

   if, furthermore, the operator $\hat{K}$ is selfadjoint, then $\left(\mathrm{L}^2(G), (\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot), \mathsf{J}_{\mathtt{m}}\right)$ is a Banach $*$-algebra;

4. *for any $f_1, f_2 \in \mathrm{L}^2(G)$, we have that*

$$f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 \in \mathcal{R}_U; \tag{95}$$

   therefore — assuming that $\|\hat{K}\| \leq 1$ — the (closed) subspace $\mathcal{R}_U$ of $\mathrm{L}^2(G)$ is a closed two-sided ideal in the Banach algebra $\left(\mathcal{B}_2(\mathcal{H}), (\cdot) \underset{\hat{K}}{\circ} (\cdot)\right)$;

5. *for any $f_1, f_2 \in \mathrm{L}^2(G)$, we have that*

$$f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 = \left(\mathrm{P}_{\mathcal{R}_U} f_1\right) \overset{U}{\underset{\hat{K}}{\star}} \left(\mathrm{P}_{\mathcal{R}_U} f_2\right); \tag{96}$$

6. *assuming that $\|\hat{K}\| \leq 1$, the application*

$$\mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \mathfrak{S}_U \hat{A} \in \mathcal{R}_U \tag{97}$$

   is an isomorphism of the Banach algebras $\left(\mathcal{B}_2(\mathcal{H}), (\cdot) \underset{\hat{K}}{\circ} (\cdot)\right)$ and $\left(\mathcal{R}_U, (\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot)\right)$.

7. *for any $f_1, f_2 \in \mathrm{L}^2(G)$, the following relation holds:*

$$\mathcal{T}_{\mathtt{m}}(g)\left(f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2\right) = \left(\mathcal{T}_{\mathtt{m}}(g) f_1\right) \overset{U}{\underset{\hat{K}_g}{\star}} \left(\mathcal{T}_{\mathtt{m}}(g) f_2\right), \quad \hat{K}_g := U(g)\, \hat{K}\, U(g)^*, \quad \forall g \in G. \tag{98}$$

It is a remarkable fact that the star product (64), and its generalization (92), which are implicitly defined via the quantization-dequantization maps associated with a square integrable representation, admit simple explicit formulae based on certain integral kernels. The task of deriving such formulae will be systematically pursued in the next section.

5 Main results: explicit formulae for star products

The aim of this section is to provide suitable expressions for the star products associated with square integrable representations that have been defined and characterized in Sect. 4. For the sake of clarity, we will split our presentation into a few subsections. In particular, in Subsect. 5.3 we will prove a simple formula for the star product $(\cdot) \overset{U}{\star} (\cdot)$ — see Theorem 2 — and from this formula we will derive various consequences, including an expression for the $\hat{K}$-deformed star product (Corollary 3). We will then show in Subsect. 5.4 — see Theorem 3 — that Theorem 2 can be actually generalized: there is a 'wide range' of possible realizations of the star product (in general, of the $\hat{K}$-deformed star product); as we will see, one for each suitably characterized right approximate identity in the H*-algebra $\mathcal{B}_2(\mathcal{H})$. Of course, we could prove Theorem 3 first, and regard Theorem 2 just as a consequence. However, the latter result can be obtained by means of a simpler procedure. So, for the reader's convenience, we prefer to prove it first.

One can find various alternative routes for getting to the main results of this section. We have tried to choose these routes in such a way to allow the reader to achieve a certain insight in 'what is going on'.

5.1 Assumptions and further notations

In the following, we will always assume that U is a square integrable projective representation (with multiplier $\mathtt{m}$) of the l.c.s.c. group G in the Hilbert space $\mathcal{H}$. We will denote, as usual, by $\hat{D}_U$ the associated Duflo-Moore operator, normalized according to a given left Haar measure μ_G on G. Recall that, if G is unimodular, then $\hat{D}_U = d_U I$, $d_U > 0$; otherwise, $\hat{D}_U$ is unbounded. We will use — often without any further explanation — the notations and the tools introduced in Sects. 2-4; in particular, we will exploit the orthogonality relations for square integrable representations and the result recalled at the end of Sect. 2.

Before starting our program, it is worth fixing a few additional notations. We will denote by ${}_{\|\cdot\|_{\mathrm{L}^2}}\lim_{n\to\infty}$ the limit of a sequence in $\mathrm{L}^2(G)$ (converging with respect to the norm $\|\cdot\|_{\mathrm{L}^2}$). Given a finite or countably infinite index set $\mathcal{N}$, we denote by ${}_{\|\cdot\|_{\mathrm{L}^2}}\sum_{n\in\mathcal{N}}$ either simply a finite sum in $\mathrm{L}^2(G)$ ($\mathcal{N}$ finite) or an infinite sum in $\mathrm{L}^2(G)$ (converging with respect to the norm $\|\cdot\|_{\mathrm{L}^2}$). Clearly, analogous meanings will be understood for the symbols ${}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{n\to\infty}$ and ${}_{\|\cdot\|_{\mathcal{B}_2}}\sum_{n\in\mathcal{N}}$ (of course, in this case the relevant space is $\mathcal{B}_2(\mathcal{H})$), or, in general, ${}_{\|\cdot\|}\sum_{n\in\mathcal{N}}$. Given a bounded operator $\hat{B}$ in $\mathcal{H}$, we can define two natural bounded operators in the Hilbert-Schmidt space $\mathcal{B}_2(\mathcal{H})$; i.e. the operators

$$\mathfrak{L}_{\hat{B}}\colon \mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \hat{B}\,\hat{A} \in \mathcal{B}_2(\mathcal{H}), \quad \mathfrak{R}_{\hat{B}}\colon \mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \hat{A}\,\hat{B} \in \mathcal{B}_2(\mathcal{H}). \tag{99}$$

It is obvious that $\mathfrak{L}_{\hat{B}}\mathfrak{R}_{\hat{B}'} = \mathfrak{R}_{\hat{B}'}\mathfrak{L}_{\hat{B}}$. In particular, given a vector $\chi \in \mathcal{H}$, we will denote by $\mathfrak{R}_{\widehat{\chi}}$ the bounded linear operator in $\mathcal{B}_2(\mathcal{H})$ defined by

$$\mathfrak{R}_{\widehat{\chi}}\colon \mathcal{B}_2(\mathcal{H}) \ni \hat{A} \mapsto \hat{A}\,\widehat{\chi} \in \mathcal{B}_2(\mathcal{H}), \tag{100}$$

where we set: $\widehat{\chi} \equiv \widehat{\chi\chi} \equiv |\chi\rangle\langle\chi|$. It is clear that — for χ nonzero and normalized — $\mathfrak{R}_{\widehat{\chi}}$ is an orthogonal projector in the Hilbert space $\mathcal{B}_2(\mathcal{H})$.

Remark 6 Let J be a complex conjugation in $\mathcal{H}$ (a selfadjoint antiunitary operator). Then, the linear map $\mathfrak{U}_J\colon \mathcal{H}\otimes\mathcal{H} \to \mathcal{B}_2(\mathcal{H})$, determined (in a consistent way) by

$$\mathfrak{U}_J\,\phi\otimes\psi = |\phi\rangle\langle J\,\psi|, \quad \forall\,\phi,\psi \in \mathcal{H}, \tag{101}$$

is a unitary operator (indeed, it is an isometry on the dense linear span generated by the separable elements of $\mathcal{H}\otimes\mathcal{H}$, and the image of this linear span is $\mathsf{FR}(\mathcal{H})$, which is dense in $\mathcal{B}_2(\mathcal{H})$). It is easy to check that $\mathfrak{U}_J(I\otimes\widehat{\chi})\mathfrak{U}_J^* = \mathfrak{R}_{\widehat{\chi}'}$, where $\widehat{\chi}' = J\widehat{\chi}J = |J\chi\rangle\langle J\chi|$. Let $\{\chi_n\}_{n\in\mathcal{N}}$ be an orthonormal basis in $\mathcal{H}$. One can always choose the complex conjugation J in such a way that $J\chi_n = \chi_n$, for any $n\in\mathcal{N}$; hence: $\mathfrak{U}_J(I\otimes\widehat{\chi}_n)\mathfrak{U}_J^* = \mathfrak{R}_{\widehat{\chi}_n}$, with $\widehat{\chi}_n \equiv |\chi_n\rangle\langle\chi_n|$. This choice of J is convenient for noting the fact that the relation ${}_{\|\cdot\|}\sum_{n\in\mathcal{N}}(I\otimes\widehat{\chi}_n\Psi) = \Psi$, $\forall\Psi\in\mathcal{H}\otimes\mathcal{H}$, is equivalent to ${}_{\|\cdot\|_{\mathcal{B}_2}}\sum_{n\in\mathcal{N}}\mathfrak{R}_{\widehat{\chi}_n}\hat{A} = \hat{A}$, $\forall\hat{A}\in\mathcal{B}_2(\mathcal{H})$. ■

Besides, given a vector χ contained in the dense linear span $\mathrm{Dom}\big(\hat{D}_U^{-1}\big)$, let $\check{\chi}$ be the linear operator in $\mathcal{H}$, of rank at most one, defined by

$$\check{\chi} := |\chi\rangle\langle\hat{D}_U^{-1}\chi|. \tag{102}$$

Then, we can consider the bounded linear operator $\mathfrak{R}_{\check{\chi}}\colon \mathcal{B}_2(\mathcal{H})\ni\hat{A}\mapsto\hat{A}\,\check{\chi}\in\mathcal{B}_2(\mathcal{H})$. Note that, if the group G is unimodular, we have: $\mathfrak{R}_{\check{\chi}} = d_U^{-1}\,\mathfrak{R}_{\widehat{\chi}}$.

Let us also introduce two integral kernels. Our formulae for star products will be based on these kernels. First — for any bounded operator $\hat{K}$ in $\mathcal{H}$ and any vector $\chi\in\mathcal{H}$, contained in the dense linear span $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$ — consider the integral kernel $\varkappa_U(\hat{K},\chi;\cdot,\cdot)\colon G\times G\to\mathbb{C}$ defined by

$$\varkappa_U(\hat{K},\chi;g,h) := \big\langle U(g)\,\hat{D}_U^{-2}\chi, \hat{K}\,U(h)\,\hat{D}_U^{-1}\chi\big\rangle = \big\langle\hat{K}^*\,U(g)\,\hat{D}_U^{-2}\chi, U(h)\,\hat{D}_U^{-1}\chi\big\rangle. \tag{103}$$

For notational convenience, we set $\varkappa_U(\chi;g,h)\equiv\varkappa_U(I,\chi;g,h) = \big\langle U(g)\,\hat{D}_U^{-2}\chi, U(h)\,\hat{D}_U^{-1}\chi\big\rangle$. Next, again for every vector χ contained in $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$, let $\kappa_U(\chi;\cdot,\cdot,\cdot)\colon G\times G\times G\to\mathbb{C}$ be the integral kernel defined by[10]

$$\kappa_U(\chi;g,h,h') := \big\langle U(g)\,\hat{D}_U^{-1}\chi, U(h)\,\hat{D}_U^{-1}\,U(h')\,\hat{D}_U^{-1}\chi\big\rangle. \tag{104}$$

Exploiting relation (11) and the fact that

$$\begin{aligned} U(h^{-1}g) = \mathtt{m}\,(h^{-1},g)\,U(h^{-1})\,U(g) &= \mathtt{m}\,(h^{-1},g)\,\mathtt{m}\,(h,h^{-1})^*\,U(h)^*\,U(g) \\ &= \mathtt{m}\,(h,h^{-1}g)^*\,U(h)^*\,U(g), \end{aligned} \tag{105}$$

we find:

$$\kappa_U(\chi;g,h,h') = \mathtt{m}\,(h,h^{-1}g)^*\Delta_G(h^{-1}g)^{\frac{1}{2}}\,\varkappa_U(\chi;h^{-1}g,h'), \quad \forall\, g,h,h'\in G. \tag{106}$$

Observe that — since $\varkappa_U(\hat{K},\chi;g,\cdot) = \mathfrak{S}_U\big(|\hat{K}^*\,U(g)\,\hat{D}_U^{-2}\chi\rangle\langle\chi|\big)^*$ — for any $g\in G$, the function $G\ni h\mapsto\varkappa_U(\hat{K},\chi;g,h)\in\mathbb{C}$ belongs to $\mathrm{L}^2(G)$. Moreover, by relation (106), for any $g,h\in G$, the function $G\ni h'\mapsto\kappa_U(\chi;g,h,h')\in\mathbb{C}$ belongs to $\mathrm{L}^2(G)$, as well.

5.2 Preliminary results

The following result will turn out to be fundamental for our purposes.

Proposition 7 *For every bounded operator $\hat{K}\in\mathcal{B}(\mathcal{H})$, for every function $f\in\mathrm{L}^2(G)$ and for every vector $\chi\in\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$, the following formula holds:*

$$\big(\mathfrak{S}_U\mathfrak{R}_{\check{\chi}}\mathfrak{L}_{\hat{K}}\mathfrak{S}_U^* f\big)(g) = \int_G \mathrm{d}\mu_G(h)\,\varkappa_U(\hat{K},\chi;g,h)\,f(h), \quad \forall_{\mu_G}\, g\in G. \tag{107}$$

[10]Recall that $\mathrm{Ran}\big(\hat{D}_U^{-1}\big)$ is a dense linear span in $\mathcal{H}$, stable under the action of the representation U; hence: $\mathrm{Dom}\big(\hat{D}_U^{-1}U(g)\,\hat{D}_U^{-1}\big) = \mathrm{Dom}\big(\hat{D}_U^{-2}\big)$, $\forall\, g\in G$.

Proof: Indeed, for every $f \in \mathrm{L}^2(G)$, we have:

$$\begin{aligned}\int_G \mathrm{d}\mu_G(h)\, \varkappa_U(\hat{K},\chi;g,h)\, f(h) &= \big\langle \mathfrak{S}_U\big(|\hat{K}^*\, U(g)\, \hat{D}_U^{-2}\chi\rangle\langle\chi|\big), f\big\rangle_{\mathrm{L}^2} \\ &= \big\langle \hat{K}^*\, |U(g)\, \hat{D}_U^{-2}\chi\rangle\langle\chi|, \mathfrak{S}_U^* f\big\rangle_{\mathcal{B}_2} \\ &= \mathsf{tr}\big(|\chi\rangle\langle U(g)\, \hat{D}_U^{-2}\chi|\, \hat{K}\,(\mathfrak{S}_U^* f)\big) \\ &= \big\langle U(g)\, \hat{D}_U^{-2}\chi, \hat{K}\,(\mathfrak{S}_U^* f)\,\chi\big\rangle, \quad \forall_{\mu_G}\, g \in G. \end{aligned} \tag{108}$$

Hence, we conclude that

$$\begin{aligned}\int_G \mathrm{d}\mu_G(h)\, \varkappa_U(\hat{K},\chi;g,h)\, f(h) &= \Big(\mathfrak{S}_U\big(\hat{K}\,(\mathfrak{S}_U^* f)\,|\chi\rangle\langle\hat{D}_U^{-1}\chi|\big)\Big)(g) \\ &= \big(\mathfrak{S}_U\, \mathfrak{R}_{\check{\chi}}\, \mathfrak{L}_{\hat{K}} \mathfrak{S}_U^* f\big)(g), \quad \forall_{\mu_G}\, g \in G. \end{aligned} \tag{109}$$

The proof of formula (107) is complete. □

At this point, in order to prove the main theorem of this section, we need to pass through three technical results. The first result (Lemma 1) will turn to be useful both in this subsection and in Subsect. 5.4. The third one (Lemma 3) 'essentially contains' the expression of the star product, already, but it requires a refinement (see Proposition 8 below) before getting to the main theorem swiftly.

Lemma 1 *For every $f \in \mathrm{L}^2(G)$ and for every $g \in G$, the following relation holds:*

$$\big(R_{\mathtt{m}}(g)\, \mathsf{J}_{\mathtt{m}}\, f\big)(h)^* = \mathtt{m}\,(h, h^{-1}g)^* \Delta_G(h^{-1}g)^{\frac{1}{2}}\, f(h^{-1}g), \tag{110}$$

$\forall_{\mu_G}\, h \in G$. Therefore, for any $f_1, f_2 \in \mathrm{L}^2(G)$ and for every $g \in G$, the function

$$G \ni h \mapsto f_1(h)\, \mathtt{m}\,(h, h^{-1}g)^* \Delta_G(h^{-1}g)^{\frac{1}{2}}\, f_2(h^{-1}g) \in \mathbb{C} \tag{111}$$

belongs to $\mathrm{L}^1(G)$ and

$$\int_G \mathrm{d}\mu_G(h)\, f_1(h)\, \mathtt{m}\,(h, h^{-1}g)^* \Delta_G(h^{-1}g)^{\frac{1}{2}}\, f_2(h^{-1}g) = \langle R_{\mathtt{m}}(g)\, \mathsf{J}_{\mathtt{m}}\, f_2, f_1\rangle_{\mathrm{L}^2}. \tag{112}$$

Proof: Recalling the definition of the representation $R_{\mathtt{m}}\colon G \to \mathcal{U}(\mathrm{L}^2(G))$ (see (14)) and of the complex conjugation $\mathsf{J}_{\mathtt{m}}\colon \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ (see (47)), we have that, for every $f \in \mathrm{L}^2(G)$ and for every $g \in G$,

$$\begin{aligned}\big(R_{\mathtt{m}}(g)\, \mathsf{J}_{\mathtt{m}}\, f\big)(h) &= \mathtt{m}\,(g, g^{-1})^*\, \mathtt{m}\,(g^{-1}, h)\, \big(\mathsf{J}_{\mathtt{m}}\, f\big)(g^{-1}h) \\ &= \mathtt{m}\,(g, g^{-1})^*\, \mathtt{m}\,(g^{-1}, h)\, \mathtt{m}\,(g^{-1}h, h^{-1}g)\, \Delta_G(g^{-1}h)^{-\frac{1}{2}}\, f(h^{-1}g)^*. \end{aligned} \tag{113}$$

Observe now that — identifying the group elements g_1, g_2, g_3 in relation (3) with g^{-1}, h and $h^{-1}g$, respectively — we have:

$$\mathtt{m}\,(g^{-1}h, h^{-1}g) = \mathtt{m}\,(g^{-1}, h)^*\, \mathtt{m}\,(g^{-1}, g)\, \mathtt{m}\,(h, h^{-1}g) = \mathtt{m}\,(g^{-1}, h)^*\, \mathtt{m}\,(g, g^{-1})\, \mathtt{m}\,(h, h^{-1}g). \tag{114}$$

From relations (113) and (114) one obtains immediately formula (110). □

Lemma 2 *For any $f_1, f_2 \in \mathrm{L}^2(G)$ and for every $\chi \in \mathrm{Dom}\big(\hat{D}_U^{-2}\big)$, the following relation holds:*

$$\int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h')\, \kappa_U(\chi;g,h,h')\, f_1(h) f_2(h') = \langle R_{\mathtt{m}}(g)\, \mathsf{J}_{\mathtt{m}}\, \mathfrak{S}_U\, \mathfrak{R}_{\check{\chi}} \mathfrak{S}_U^* f_2, f_1\rangle_{\mathrm{L}^2}. \tag{115}$$

Proof: Taking into account (106), by relation (107) — with $\hat{K} = I$ — we obtain:

$$\begin{aligned}\int_G \mathrm{d}\mu_G(h')\,\kappa_U(\chi;g,h,h')\,f_2(h') &= \mathtt{m}\,(h,h^{-1}g)^*\Delta_G(h^{-1}g)^{\frac{1}{2}}\int_G \mathrm{d}\mu_G(h')\,\varkappa_U(\chi;h^{-1}g,h')\,f_2(h') \\ &= \mathtt{m}\,(h,h^{-1}g)^*\Delta_G(h^{-1}g)^{\frac{1}{2}}\big(\mathfrak{S}_U\mathfrak{R}_{\check{\chi}}\mathfrak{S}_U^* f_2\big)(h^{-1}g). \end{aligned} \tag{116}$$

At this point, relation (115) is a straightforward consequence of Lemma 1. □

Lemma 3 *Let χ be a vector belonging to $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$. Then, for every $\phi_1 \in \mathcal{H}$, and for any $\psi_1,\psi_2,\phi_2 \in \mathrm{Dom}\big(\hat{D}_U^{-1}\big)$ — setting, as usual, $\widehat{\phi_j\psi_j} \equiv |\phi_j\rangle\langle\psi_j|$, $j=1,2$ — we have:*

$$\begin{aligned}\Big(\mathfrak{S}_U\mathfrak{R}_{\widehat{\chi}}\big(\widehat{\phi_1\psi_1}\,\widehat{\phi_2\psi_2}\big)\Big)(g) &= \int_G \mathrm{d}\mu_G(h)\int_G \mathrm{d}\mu_G(h')\,\kappa_U(\chi;g,h,h') \\ &\quad\times\big(\mathfrak{S}_U\widehat{\phi_1\psi_1}\big)(h)\,\big(\mathfrak{S}_U\widehat{\phi_2\psi_2}\big)(h'),\quad \forall_{\mu_G}\, g\in G. \end{aligned} \tag{117}$$

Proof: First observe that

$$\begin{aligned}\int_G \mathrm{d}\mu_G(h')\,\kappa_U(\chi;g,h,h')\,\big(\mathfrak{S}_U\widehat{\phi_2\psi_2}\big)(h') &= \int_G \mathrm{d}\mu_G(h')\,\big\langle \hat{D}_U^{-1}\,U(h)^*\,U(g)\,\hat{D}_U^{-1}\chi, U(h')\,\hat{D}_U^{-1}\chi\big\rangle \\ &\quad\times\big\langle U(h')\,\hat{D}_U^{-1}\psi_2,\phi_2\big\rangle \\ &= \big\langle\psi_2,\chi\big\rangle\,\langle \hat{D}_U^{-1}\,U(h)^*\,U(g)\,\hat{D}_U^{-1}\chi,\phi_2\big\rangle \\ &= \langle\psi_2,\chi\rangle\,\big\langle U(g)\,\hat{D}_U^{-1}\chi, U(h)\,\hat{D}_U^{-1}\phi_2\big\rangle, \end{aligned} \tag{118}$$

$\forall\, h,g\in G$, where we have used the fact that ϕ_2 is contained in $\mathrm{Dom}\big(\hat{D}_U^{-1}\big)$. Then, exploiting relation (118) and the fact that

$$\int_G \mathrm{d}\mu_G(h)\,\big\langle U(g)\,\hat{D}_U^{-1}\chi, U(h)\hat{D}_U^{-1}\phi_2\big\rangle\,\big\langle U(h)\,\hat{D}_U^{-1}\psi_1,\phi_1\big\rangle = \langle\psi_1,\phi_2\rangle\,\big\langle U(g)\,\hat{D}_U^{-1}\chi,\phi_1\big\rangle \tag{119}$$

— note that $\big\langle U(h)\,\hat{D}_U^{-1}\psi_1,\phi_1\big\rangle = \big(\mathfrak{S}_U\widehat{\phi_1\psi_1}\big)(h)$ — we find:

$$\begin{aligned}&\int_G \mathrm{d}\mu_G(h)\int_G \mathrm{d}\mu_G(h')\,\kappa_U(\chi;g,h,h')\,\big(\mathfrak{S}_U\widehat{\phi_1\psi_1}\big)(h)\,\big(\mathfrak{S}_U\widehat{\phi_2\psi_2}\big)(h') \\ &= \langle\psi_2,\chi\rangle\int_G \mathrm{d}\mu_G(h)\,\big\langle U(g)\,\hat{D}_U^{-1}\chi, U(h)\hat{D}_U^{-1}\phi_2\big\rangle\,\big(\mathfrak{S}_U\widehat{\phi_1\psi_1}\big)(h) \\ &= \langle\psi_2,\chi\rangle\,\langle\psi_1,\phi_2\rangle\,\big\langle U(g)\,\hat{D}_U^{-1}\chi,\phi_1\big\rangle = \mathfrak{S}_U\big(\widehat{\phi_1\psi_1}\,\widehat{\phi_2\psi_2}\,\widehat{\chi}\big)(g). \end{aligned} \tag{120}$$

The proof is complete. □

As anticipated, the following result can be regarded as a generalization of Lemma 3. It will allow us to prove the main result of this section in a straightforward and transparent way.

Proposition 8 *Let χ be a vector contained in $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$. Then, for any $f_1,f_2\in\mathrm{L}^2(G)$, the following formula holds:*

$$\mathfrak{S}_U\mathfrak{R}_{\widehat{\chi}}\big((\mathfrak{S}_U^* f_1)(\mathfrak{S}_U^* f_2)\big) = \int_G \mathrm{d}\mu_G(h)\int_G \mathrm{d}\mu_G(h')\,\kappa_U(\chi;\cdot,h,h')\,f_1(h)\,f_2(h'). \tag{121}$$

Proof: By Lemma 3, relation (121) holds for any pair of functions f_1,f_2 belonging to the linear span $\mathfrak{S}_U(\mathsf{FR}^{|\rangle\langle|}(\mathcal{H};U))$ (see (27)), which is dense in $\mathcal{R}_U$. Moreover — since $\mathrm{Ker}\big(\mathfrak{S}_U^*\big) = \mathcal{R}_U^\perp$,

and $\mathcal{R}_U$ is an invariant subspace for the complex conjugation $\mathsf{J}_{\mathtt{m}}$ and for the representation $R_{\mathtt{m}}$ — for any pair of functions $f_1, f_2 \in \mathrm{L}^2(G)$, of which at least one is contained in $\mathcal{R}_U^{\perp}$, we have:

$$\langle R_{\mathtt{m}}(g)\mathsf{J}_{\mathtt{m}}\,\mathfrak{S}_U\mathfrak{R}_{\check{\chi}}\mathfrak{S}_U^* f_2, f_1\rangle_{\mathrm{L}^2} = 0. \tag{122}$$

Thus, if f_1 and/or f_2 is contained in $\mathcal{R}_U^{\perp}$, recalling relation (115) we conclude that

$$\int_G \mathrm{d}\mu_G(h)\int_G \mathrm{d}\mu_G(h')\,\kappa_U(\chi;\cdot,h,h')\,f_1(h)f_2(h') = 0. \tag{123}$$

Therefore, relation (121) is satisfied by f_1, f_2 in the dense linear span $\mathfrak{S}_U(\mathsf{FR}^{|\rangle\langle|}(\mathcal{H};U)) + \mathcal{R}_U^{\perp}$. In the case where the Hilbert space $\mathcal{H}$ is finite-dimensional (hence, G is unimodular), this linear span actually coincides with $\mathrm{L}^2(G)$ itself and the proof is complete.

Let us assume, instead, that $\dim(\mathcal{H}) = \infty$, and let us prove relation (121) for a generic pair of functions in $\mathrm{L}^2(G)$. To this aim, consider first a pair of functions f_1, f_2 of this kind: f_1 is an arbitrary function contained in the dense linear span $\mathfrak{S}_U(\mathsf{FR}^{|\rangle\langle|}(\mathcal{H};U)) + \mathcal{R}_U^{\perp}$, and f_2 any function belonging to $\mathrm{L}^2(G)$. Next, take a sequence of functions $\{f_{2;n}\}_{n\in\mathbb{N}} \subset \mathrm{L}^2(G)$, contained in $\mathfrak{S}_U(\mathsf{FR}^{|\rangle\langle|}(\mathcal{H};U)) + \mathcal{R}_U^{\perp}$ and converging (with respect to the norm $\|\cdot\|_{\mathrm{L}^2}$) to f_2. Then, we have:

$$\|\cdot\|_{\mathrm{L}^2}\lim_{n\to\infty}\mathfrak{S}_U\mathfrak{R}_{\widehat{\chi}}\big((\mathfrak{S}_U^* f_1)(\mathfrak{S}_U^* f_{2;n})\big) = \mathfrak{S}_U\mathfrak{R}_{\widehat{\chi}}\big((\mathfrak{S}_U^* f_1)(\mathfrak{S}_U^* f_2)\big). \tag{124}$$

On the other hand, by the first part of the proof and by Lemma 2, we have that

$$\begin{aligned}
\lim_{n\to\infty}\Big(\mathfrak{S}_U\mathfrak{R}_{\widehat{\chi}}\big((\mathfrak{S}_U^* f_1)(\mathfrak{S}_U^* f_{2;n})\big)\Big)(g) &= \lim_{n\to\infty}\int_G \mathrm{d}\mu_G(h)\int_G \mathrm{d}\mu_G(h')\,\kappa_U(\chi;g,h,h')\,f_1(h)f_{2;n}(h')\\
&= \lim_{n\to\infty}\langle R_{\mathtt{m}}(g)\mathsf{J}_{\mathtt{m}}\,\mathfrak{S}_U\mathfrak{R}_{\check{\chi}}\mathfrak{S}_U^* f_{2;n}, f_1\rangle_{\mathrm{L}^2}\\
&= \langle R_{\mathtt{m}}(g)\mathsf{J}_{\mathtt{m}}\,\mathfrak{S}_U\mathfrak{R}_{\check{\chi}}\mathfrak{S}_U^* f_2, f_1\rangle_{\mathrm{L}^2}\\
&= \int_G \mathrm{d}\mu_G(h)\int_G \mathrm{d}\mu_G(h')\,\kappa_U(\chi;g,h,h')\,f_1(h)f_2(h').
\end{aligned} \tag{125}$$

From relations (124) and (125) it descends that formula (121) holds true for any pair of functions $f_1 \in \big(\mathfrak{S}_U(\mathsf{FR}^{|\rangle\langle|}(\mathcal{H};U)) + \mathcal{R}_U^{\perp}\big)$ and $f_2 \in \mathrm{L}^2(G)$. At this point, using this result and a density argument analogous to the one adopted for obtaining it, one proves relation (121) for a generic pair of functions in $\mathrm{L}^2(G)$. □

Remark 7 One can arrive at formula (121) by various alternative routes. For instance, one can derive it from Lemma 2 using the intertwining relations (46) and (49). This way will be adopted for proving Theorem 3. The above proof offers the advantage of a direct computation. Another way is to prove that, for every $f \in \mathrm{L}^2(G)$,

$$\int_G \mathrm{d}\mu_G(h)\,\kappa_U(\chi;g,h,(\cdot))\,f(h) = \mathfrak{S}_U\Big(\big((\mathfrak{S}_U^* f)^*\,|U(g)\hat{D}_U^{-1}\chi\rangle\langle\chi|\Big)^*, \tag{126}$$

and from this relation deduce that the function $\int_G \mathrm{d}\mu_G(h')\int_G \mathrm{d}\mu_G(h)\,\kappa_U(\chi;\cdot,h,h')\,f_1(h)\,f_2(h')$ is equal to the function on the l.h.s. of (121), for all $f_1, f_2 \in \mathrm{L}^2(G)$. Observe that this shows, in particular, that *the iterated integrals on the r.h.s. of* (121) *can be permuted.* ■

5.3 Formulae for star products

We are now ready to prove the theorem that can be regarded as the main result of this section. It provides a simple expression for the star product associated with the square integrable projective representation U.

Theorem 2 *Let $\{\chi_n\}_{n\in\mathcal{N}}$ be an orthonormal basis in $\mathcal{H}$, contained in the dense linear span $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$. Then, for any $f_1, f_2 \in \mathrm{L}^2(G)$, the following formula holds:*

$$f_1 \overset{U}{\star} f_2 = {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h')\, \kappa_U(\chi_n;\cdot,h,h')\, f_1(h) f_2(h'), \tag{127}$$

where the integral kernel $\kappa_U(\chi_n;\cdot,\cdot,\cdot)\colon G\times G\times G \to \mathbb{C}$ is defined by (104), i.e.

$$\kappa_U(\chi_n; g, h, h') := \big\langle U(g)\, \hat{D}_U^{-1}\chi_n, U(h)\, \hat{D}_U^{-1}\, U(h')\, \hat{D}_U^{-1}\chi_n\big\rangle. \tag{128}$$

Proof: In order to prove formula (127) we can exploit relation (121) and the fact that

$$ {}_{\|\cdot\|_{\mathcal{B}_2}}\!\sum_{n\in\mathcal{N}} \mathfrak{R}_{\widehat{\chi}_n} \hat{A} = \hat{A}, \quad \forall \hat{A} \in \mathcal{B}_2(\mathcal{H}), \tag{129}$$

where $\widehat{\chi}_n \equiv |\chi_n\rangle\langle\chi_n|$; see Remark 6. Indeed, for any $f_1, f_2 \in \mathrm{L}^2(G)$, we have:

$$\begin{aligned} {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h')\, \kappa_U(\chi_n;\cdot,h,h')\, f_1(h) f_2(h') &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \mathfrak{S}_U\, \mathfrak{R}_{\widehat{\chi}_n}\big((\mathfrak{S}_U^* f_1)(\mathfrak{S}_U^* f_2)\big) \\ &= \mathfrak{S}_U\; {}_{\|\cdot\|_{\mathcal{B}_2}}\!\sum_{n\in\mathcal{N}} \mathfrak{R}_{\widehat{\chi}_n}\big((\mathfrak{S}_U^* f_1)(\mathfrak{S}_U^* f_2)\big) \\ &= \mathfrak{S}_U\big((\mathfrak{S}_U^* f_1)(\mathfrak{S}_U^* f_2)\big). \end{aligned} \tag{130}$$

By definition, the last member of (130) is equal to $f_1 \overset{U}{\star} f_2$. □

Remark 8 Taking into account the last assertion of Remark 7, we conclude that *the iterated integrals on the r.h.s. of formula* (127) *can be permuted.* ■

Remark 9 One can readily derive from formula (127) various alternative expressions for the star product; in particular:

$$\begin{aligned} f_1 \overset{U}{\star} f_2 &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h)\, f_1(h)\, \mathtt{m}\,(h, h^{-1}(\cdot))^* \Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \int_G \mathrm{d}\mu_G(h')\, \varkappa_U(\chi_n; h^{-1}(\cdot), h')\, f_2(h') \\ &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h)\, f_1((\cdot)h)\, \mathtt{m}\,((\cdot)h, h^{-1})^* \Delta_G(h^{-1})^{\frac{1}{2}} \int_G \mathrm{d}\mu_G(h')\, \varkappa_U(\chi_n; h^{-1}, h')\, f_2(h') \\ &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h)\, f_1(h^{-1})\, \mathtt{m}\,(h^{-1}, h(\cdot))^* \Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \\ &\times \int_G \mathrm{d}\mu_G(h')\, \varkappa_U(\chi_n; h(\cdot), h')\, f_2(h'), \quad \forall f_1, f_2 \in \mathrm{L}^2(G). \end{aligned} \tag{131}$$

The first expression is obtained by relation (106); then, by the change of variables $h \mapsto gh$ and $h \mapsto h^{-1}$ (recall that $\int_G \mathrm{d}\mu_G(h)\, f(h) = \int_G \mathrm{d}\mu_G(h)\, \Delta_G(h)^{-1} f(h^{-1})$), from the first expression one obtains the other two. ■

Remark 10 It is rather boring, but straightforward, to check that, for every $\chi \in \mathrm{Dom}\big(\hat{D}_U^{-2}\big)$,

$$\begin{aligned} &\int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h')\, \kappa_U(\chi;\cdot,h,h')\, \big(\mathcal{T}_{\mathtt{m}}(g)\, f_1\big)(h)\, \big(\mathcal{T}_{\mathtt{m}}(g)\, f_2\big)(h') \\ &= \mathcal{T}_{\mathtt{m}}(g) \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h')\, \kappa_U(U(g^{-1})\,\chi;\cdot,h,h')\, f_1(h) f_2(h'). \end{aligned} \tag{132}$$

Since formula (127) does not depend on a specific choice of the orthonormal basis $\{\chi_n\}_{n\in\mathcal{N}}$ contained in $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$ (recall that this dense linear span is stable with respect to U), relation (132) confirms the fact that $\big(\mathcal{T}_{\mathtt{m}}(g)\, f_1\big) \stackrel{U}{\star} \big(\mathcal{T}_{\mathtt{m}}(g)\, f_2\big) = \mathcal{T}_{\mathtt{m}}(g) \left(f_1 \stackrel{U}{\star} f_2\right)$, $\forall\, f_1, f_2 \in \mathrm{L}^2(G)$ — see (81)— i.e. the equivariance of the star product with respect to the representation $\mathcal{T}_{\mathtt{m}}$. ■

Theorem 2 has various implications. First of all, it is remarkable that, in the case where G is unimodular, the star product associated with the representation U admits a simple alternative expression.

Corollary 1 *Suppose that the l.c.s.c. group G is unimodular. Then, for any $f_1, f_2 \in \mathrm{L}^2(G)$, we have:*

$$\begin{aligned}
\left(f_1 \stackrel{U}{\star} f_2\right)(g) &= d_U^{-1} \int_G \mathrm{d}\mu_G(h)\, f_1(h)\, \mathtt{m}\,(h, h^{-1}g)^* \big(\mathrm{P}_{\mathcal{R}_U} f_2\big)(h^{-1}g) \\
&= d_U^{-1} \int_G \mathrm{d}\mu_G(h)\, \big(\mathrm{P}_{\mathcal{R}_U} f_1\big)(h)\, \mathtt{m}\,(h, h^{-1}g)^* f_2(h^{-1}g) \\
&= d_U^{-1} \int_G \mathrm{d}\mu_G(h)\, \big(\mathrm{P}_{\mathcal{R}_U} f_1\big)(h)\, \mathtt{m}\,(h, h^{-1}g)^* \big(\mathrm{P}_{\mathcal{R}_U} f_2\big)(h^{-1}g), \quad \forall_{\mu_G}\, g \in G. \quad (133)
\end{aligned}$$

Therefore, for any $f_1, f_2 \in \mathcal{R}_U$, the following formula holds:

$$\left(f_1 \stackrel{U}{\star} f_2\right)(g) = d_U^{-1} \int_G \mathrm{d}\mu_G(h)\, f_1(h)\, \mathtt{m}\,(h, h^{-1}g)^* f_2(h^{-1}g), \quad \forall_{\mu_G}\, g \in G. \tag{134}$$

Proof: Let f_1, f_2 be functions in $\mathrm{L}^2(G)$. Then — using formula (127), relation (115) and the fact that $\mathfrak{R}_{\check{\chi}} = d_U^{-1}\, \mathfrak{R}_{\widehat{\chi}}$ (since G is unimodular) — we have:

$$\begin{aligned}
f_1 \stackrel{U}{\star} f_2 &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h')\, \kappa_U(\chi_n; \cdot, h, h')\, f_1(h) f_2(h') \\
&= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \langle R_{\mathtt{m}}(\cdot)\, \mathsf{J}_{\mathtt{m}}\, \mathfrak{S}_U \mathfrak{R}_{\check{\chi}_n} \mathfrak{S}_U^* f_2, f_1\rangle_{\mathrm{L}^2} \\
&= d_U^{-1}\, {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \langle R_{\mathtt{m}}(\cdot)\, \mathsf{J}_{\mathtt{m}}\, \mathfrak{S}_U \mathfrak{R}_{\widehat{\chi}_n} \mathfrak{S}_U^* f_2, f_1\rangle_{\mathrm{L}^2}\,. \quad (135)
\end{aligned}$$

On the other hand — by virtue of the continuity of the scalar product in $\mathrm{L}^2(G)$ and of the boundedness of the operators $R_{\mathtt{m}}(g)$, $\mathsf{J}_{\mathtt{m}}$ and $\mathfrak{S}_U$, and exploiting relations (129) and, then, (110) with $(\Delta_G \equiv 1)$ — we also have that

$$\begin{aligned}
\sum_{n\in\mathcal{N}} \langle R_{\mathtt{m}}(g)\, \mathsf{J}_{\mathtt{m}}\, \mathfrak{S}_U \mathfrak{R}_{\widehat{\chi}_n} \mathfrak{S}_U^* f_2, f_1\rangle_{\mathrm{L}^2} &= \langle R_{\mathtt{m}}(g)\, \mathsf{J}_{\mathtt{m}}\, \mathfrak{S}_U \mathfrak{S}_U^* f_2, f_1\rangle_{\mathrm{L}^2} \\
&= \big\langle R_{\mathtt{m}}(g)\, \mathsf{J}_{\mathtt{m}}\, \mathrm{P}_{\mathcal{R}_U} f_2, f_1\big\rangle_{\mathrm{L}^2} \\
&= \int_G \mathrm{d}\mu_G(h)\, f_1(h)\, \mathtt{m}\,(h, h^{-1}g)^* \big(\mathrm{P}_{\mathcal{R}_U} f_2\big)(h^{-1}g). \quad (136)
\end{aligned}$$

Relations (135) and (136) imply that the first of equations (133) holds true; the other two are obtained using the fact that $\mathrm{P}_{\mathcal{R}_U}$ is a projector satisfying $R_{\mathtt{m}}(g)\, \mathsf{J}_{\mathtt{m}}\, \mathrm{P}_{\mathcal{R}_U} = \mathrm{P}_{\mathcal{R}_U} R_{\mathtt{m}}(g)\, \mathsf{J}_{\mathtt{m}}$. □

Remark 11 We stress that the particularly simple formula (134) — differently from formula (127) — holds for any pair of functions $f_1, f_2 \in \mathrm{L}^2(G)$ of which *at least one* belongs to the (closed) subspace $\mathcal{R}_U$ of $\mathrm{L}^2(G)$, which is the canonical ideal of the H*-algebra $\mathcal{A}_U$, see Proposition 5. The r.h.s. of (134) is a 'twisted convolution' generalizing the standard twisted convolution [20] that appears in the case where G is the group of translations on phase space and U is the projective representation (21) (we will examine this case in Sect. 6). ■

Let us derive another consequence of Theorem 2. In the case where the group G is compact (hence, unimodular), there is a precise link between the convolution product in $\mathrm{L}^2(G)$ [22] and the star products associated with a realization $\check{G}$ of the unitary dual of G.

Corollary 2 *Suppose that the l.c.s.c. group G is compact and that the Haar measure μ_G is normalized as usual for compact groups, i.e. that $\mu_G(G) = 1$. Then, for any $f_1, f_2 \in \mathrm{L}^2(G)$, the following formula holds:*

$$\mathrm{L}^2(G) \ni \int_G \mathrm{d}\mu_G(h)\, f_1(h)\, f_2(h^{-1}(\cdot)) = {\scriptstyle \|\cdot\|_{\mathrm{L}^2}}\!\!\sum_{U\in\check{G}} \delta(U)^{-\frac{1}{2}} \left(f_1 \overset{U}{\star} f_2\right). \tag{137}$$

Proof: As is well known, since G is compact, the convolution of any pair of functions in $\mathrm{L}^2(G)$ is again a function belonging to $\mathrm{L}^2(G)$. Moreover, from relation (36) it follows that ${\scriptstyle \|\cdot\|_{\mathrm{L}^2}}\sum_{U\in\check{G}} \mathrm{P}_{\mathcal{R}_U} f = f$, $\forall\, f \in \mathrm{L}^2(G)$; hence — denoting by R the left regular representation of G and by J the complex conjugation

$$\mathrm{L}^2(G) \ni f \mapsto f((\cdot)^{-1})^* \in \mathrm{L}^2(G) \tag{138}$$

— for any $f_1, f_2 \in \mathrm{L}^2(G)$ we have:

$$\begin{aligned}
\int_G \mathrm{d}\mu_G(h)\, f_1(h)\, f_2(h^{-1}g) &= \int_G \mathrm{d}\mu_G(h) \Big({\scriptstyle \|\cdot\|_{\mathrm{L}^2}}\!\!\sum_{U\in\check{G}} \mathrm{P}_{\mathcal{R}_U} f_1\Big)(h)\, f_2(h^{-1}g) \\
&= \big\langle R(g)\mathsf{J} f_2, {\scriptstyle \|\cdot\|_{\mathrm{L}^2}}\!\!\sum_{U\in\check{G}} \mathrm{P}_{\mathcal{R}_U} f_1 \big\rangle_{\mathrm{L}^2} \\
&= \sum_{U\in\check{G}} \big\langle R(g)\mathsf{J} f_2, \mathrm{P}_{\mathcal{R}_U} f_1 \big\rangle_{\mathrm{L}^2} \\
&= \sum_{U\in\check{G}} \int_G \mathrm{d}\mu_G(h) \left(\mathrm{P}_{\mathcal{R}_U} f_1\right)(h)\, f_2(h^{-1}g),
\end{aligned} \tag{139}$$

for all $g \in G$. On the other hand, by Corollary 1 we have that

$$\int_G \mathrm{d}\mu_G(h) \left(\mathrm{P}_{\mathcal{R}_U} f_1\right)(h)\, f_2(h^{-1}(\cdot)) = \delta(U)^{-\frac{1}{2}} \left(f_1 \overset{U}{\star} f_2\right), \quad \forall\, U \in \check{G}, \tag{140}$$

where we recall that $\delta(U)^{-\frac{1}{2}} = d_U$. Moreover, by relations (79) and (76), for any $f_1, f_2 \in \mathrm{L}^2(G)$ we obtain the following estimate:

$$\begin{aligned}
\sum_{U\in\check{G}} \delta(U)^{-1} \left\|f_1 \overset{U}{\star} f_2\right\|^2_{\mathrm{L}^2} &= \sum_{U\in\check{G}} \delta(U)^{-1} \left\|\left(\mathrm{P}_{\mathcal{R}_U} f_1\right) \overset{U}{\star} \left(\mathrm{P}_{\mathcal{R}_U} f_2\right)\right\|^2_{\mathrm{L}^2} \\
&\le \sum_{U\in\check{G}} \delta(U)^{-1} \left\|\mathrm{P}_{\mathcal{R}_U} f_1\right\|^2_{\mathrm{L}^2} \left\|\mathrm{P}_{\mathcal{R}_U} f_2\right\|^2_{\mathrm{L}^2} \\
&\le \sum_{U\in\check{G}} \left\|\mathrm{P}_{\mathcal{R}_U} f_1\right\|^2_{\mathrm{L}^2} \left\|\mathrm{P}_{\mathcal{R}_U} f_2\right\|^2_{\mathrm{L}^2} \le \|f_1\|^2_{\mathrm{L}^2} \|f_2\|^2_{\mathrm{L}^2}.
\end{aligned} \tag{141}$$

Hence, taking into account (78), we see that ${\scriptstyle \|\cdot\|_{\mathrm{L}^2}}\sum_{U\in\check{G}} \delta(U)^{-\frac{1}{2}} \left(f_1 \overset{U}{\star} f_2\right)$ is a well defined element of $\mathrm{L}^2(G)$ and, by (140),

$${\scriptstyle \|\cdot\|_{\mathrm{L}^2}}\!\!\sum_{U\in\check{G}} \int_G \mathrm{d}\mu_G(h) \left(\mathrm{P}_{\mathcal{R}_U} f_1\right)(h)\, f_2(h^{-1}(\cdot)) = {\scriptstyle \|\cdot\|_{\mathrm{L}^2}}\!\!\sum_{U\in\check{G}} \delta(U)^{-\frac{1}{2}} \left(f_1 \overset{U}{\star} f_2\right). \tag{142}$$

At this point, relations (139) and (142) imply that formula (137) holds true. □

We will now prove that it is possible to achieve a simple expression of the $\hat{K}$-deformed star product associated with the representation U, for every bounded operator $\hat{K} \in \mathcal{B}(\mathcal{H})$. Although this result is more general than Theorem 2 — which corresponds to the case where $\hat{K} = I$ — we will derive it as a consequence of formula (127) for the star product. To this aim, it is useful to observe that, by the definition of the $\hat{K}$-deformed star product and the fact that $\mathfrak{S}_U^*\mathfrak{S}_U = I$, we have:

$$\begin{aligned} f_1 \underset{\hat{K}}{\overset{U}{\star}} f_2 &:= \mathfrak{S}_U\big(\mathfrak{S}_U^* f_1\, \hat{K}\, \mathfrak{S}_U^* f_2\big) \\ &= \mathfrak{S}_U\Big(\mathfrak{S}_U^* f_1\, \mathfrak{S}_U^*\big(\mathfrak{S}_U(\hat{K}\, \mathfrak{S}_U^* f_2)\big)\Big) = f_1 \overset{U}{\star} \big(\mathfrak{S}_U(\hat{K}\, \mathfrak{S}_U^* f_2)\big). \end{aligned} \tag{143}$$

Moreover, for every bounded operator $\hat{K}$ in $\mathcal{H}$ and for every vector χ contained in $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$, let us define an integral kernel $\kappa_U(\hat{K}, \chi; \cdot, \cdot, \cdot)\colon G \times G \times G \to \mathbb{C}$ by setting:

$$\begin{aligned} \kappa_U(\hat{K}, \chi; g, h, h') &:= \langle \hat{D}_U^{-1}\, U(h)^*\, U(g)\, \hat{D}_U^{-1}\chi, \hat{K}\, U(h')\, \hat{D}_U^{-1}\chi\rangle \\ &= \mathtt{m}\,(h, h^{-1}g)^* \Delta_G(h^{-1}g)^{\frac{1}{2}}\, \varkappa_U(\hat{K}, \chi; h^{-1}g, h'). \end{aligned} \tag{144}$$

Comparing this definition with (104), it is clear that $\kappa_U(\chi; g, h, h') \equiv \kappa_U(I, \chi; g, h, h')$.

Corollary 3 *Let $\hat{K}$ be a bounded operator in $\mathcal{H}$ and $\{\chi_n\}_{n\in\mathcal{N}}$ an orthonormal basis contained in the dense linear span $\mathrm{Dom}\big(\hat{D}_U^{-2}\big)$. Then, for any $f_1, f_2 \in \mathrm{L}^2(G)$, the following formula holds:*

$$f_1 \underset{\hat{K}}{\overset{U}{\star}} f_2 = {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h')\, \kappa_U(\hat{K}, \chi_n; \cdot, h, h')\, f_1(h) f_2(h'). \tag{145}$$

Proof: Taking into account relation (143), we can apply formula (127) for the (standard) star product, and next we use relation (115), thus getting

$$\begin{aligned} f_1 \underset{\hat{K}}{\overset{U}{\star}} f_2 &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h')\, \kappa_U(\chi_n; \cdot, h, h')\, f_1(h) \big(\mathfrak{S}_U(\hat{K}\, \mathfrak{S}_U^* f_2)\big)(h') \\ &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \big\langle R_{\mathtt{m}}(\cdot)\, \mathsf{J}_{\mathtt{m}}\, \mathfrak{S}_U \mathfrak{R}_{\check{\chi}_n} \mathfrak{S}_U^*\big(\mathfrak{S}_U(\hat{K}\, \mathfrak{S}_U^* f_2)\big), f_1\big\rangle_{\mathrm{L}^2} \\ &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \big\langle R_{\mathtt{m}}(\cdot)\, \mathsf{J}_{\mathtt{m}}\, \big(\mathfrak{S}_U \mathfrak{R}_{\check{\chi}_n}(\hat{K}\, \mathfrak{S}_U^* f_2)\big), f_1\big\rangle_{\mathrm{L}^2}. \end{aligned} \tag{146}$$

From (146), by virtue of relations (112) and (107), it follows that

$$\begin{aligned} f_1 \underset{\hat{K}}{\overset{U}{\star}} f_2 &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h)\, f_1(h)\, \mathtt{m}\,(h, h^{-1}g)^* \Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \big(\mathfrak{S}_U \mathfrak{R}_{\check{\chi}_n} \mathfrak{L}_{\hat{K}}\, \mathfrak{S}_U^* f_2\big)(h^{-1}(\cdot)) \\ &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h)\, f_1(h)\, \mathtt{m}\,(h, h^{-1}(\cdot))^* \Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \\ &\times \int_G \mathrm{d}\mu_G(h')\, \varkappa_U(\hat{K}, \chi_n; h^{-1}(\cdot), h')\, f_2(h') \\ &= {}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathcal{N}} \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h')\, \kappa_U(\hat{K}, \chi_n; \cdot, h, h')\, f_1(h) f_2(h'), \end{aligned} \tag{147}$$

where the last member is obtained by (144).

The proof is complete. □

Formula (145) assumes a remarkably simple form in the special case where the carrier Hilbert space $\mathcal{H}$ of the representation U is finite-dimensional (so that the l.c.s.c. group G must be unimodular; see the last assertion of Remark 1).

Corollary 4 *Suppose that the Hilbert space $\mathcal{H}$, where the square integrable representation U acts, is finite-dimensional. Then, for any pair of functions $f_1, f_2 \in \mathrm{L}^2(G)$, the following formula holds:*

$$f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 = d_U^{-3} \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h') \; \mathtt{tr}(U(\cdot)^* \, U(h) \, \hat{K} \, U(h')) \, f_1(h) f_2(h') . \tag{148}$$

Proof: If $\mathcal{H}$ is finite-dimensional, then on the r.h.s. of formula (145) we have a finite sum and $\hat{D}_U^{-1} = d_U^{-1} I$; therefore:

$$\Big(f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2\Big)(g) = \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h') \sum_{n \in \mathcal{N}} \kappa_U(\hat{K}, \chi_n; g, h, h') \, f_1(h) f_2(h') , \quad \forall_{\mu_G} g \in G, \tag{149}$$

where

$$\begin{aligned} \sum_{n \in \mathcal{N}} \kappa_U(\hat{K}, \chi_n; g, h, h') &= d_U^{-3} \sum_{n \in \mathcal{N}} \langle \chi_n, U(g)^* \, U(h) \, \hat{K} \, U(h') \, \chi_n \rangle \\ &= d_U^{-3} \, \mathtt{tr}(U(g)^* \, U(h) \, \hat{K} \, U(h')) , \end{aligned} \tag{150}$$

by definition of the trace. □

Remark 12 Assume that G is a compact — in particular, a finite — group and U is a (irreducible) unitary representation. In this case, formula (148) reads:

$$f_1 \overset{U}{\star} f_2 = \delta(U)^{\frac{3}{2}} \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h') \; \mathtt{C}_U\big((\cdot)^{-1} h h'\big) \, f_1(h) f_2(h') . \tag{151}$$

where the function $\mathtt{C}_U \colon G \to \mathbb{C}$ is the character of the finite-dimensional representation U; i.e. $\mathtt{C}_U(g) := \mathtt{tr}(U(g))$. Then, since $\mathfrak{S}_U I = \delta(U)^{\frac{1}{2}} \, \mathtt{C}_U\big((\cdot)^{-1}\big)$, the obvious equation

$$\big(\mathfrak{S}_U I\big) \overset{U}{\star} \big(\mathfrak{S}_U I\big) = \mathfrak{S}_U I \tag{152}$$

translates into the following relation for the character $\mathtt{C}_U$:

$$\mathtt{C}_U(g) = \delta(U)^2 \int_G \mathrm{d}\mu_G(h) \int_G \mathrm{d}\mu_G(h') \; \mathtt{C}_U\big(g h h'\big) \, \mathtt{C}_U\big(h^{-1}\big) \, \mathtt{C}_U\big((h')^{-1}\big) . \tag{153}$$

Thus, we recover results previously found in ref. [16]. ■

5.4 A generalization of Theorem 2

As anticipated, Theorem 2 can be further generalized. This generalization is based on the notion of *right approximate identity* in a Banach algebra — in particular, in a H*-algebra. We will say that a sequence $\{\hat{T}_n\}_{n \in \mathbb{N}} \subset \mathcal{B}_2(\mathcal{H})$ is a right approximate identity in the H*-algebra $\mathcal{B}_2(\mathcal{H})$ if ${}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{n \to \infty} \hat{A} \, \hat{T}_n = \hat{A}$, for all $\hat{A} \in \mathcal{B}_2(\mathcal{H})$; otherwise stated, if the sequence $\big\{\mathfrak{R}_{\hat{T}_n}\big\}_{n \in \mathbb{N}}$ of bounded operators in the Hilbert space $\mathcal{B}_2(\mathcal{H})$ is strongly convergent to the identity. As is well known, the H*-algebra $\mathcal{B}_2(\mathcal{H})$ admits an identity if and only if $\mathcal{H}$ is finite-dimensional, but it always admits a right approximate identity. For instance, in the case where $\mathcal{H}$ is infinite-dimensional, for every orthonormal basis $\{\chi_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ the sequence $\{\sum_{k=1}^n |\chi_k\rangle\langle\chi_k|\}_{n \in \mathbb{N}}$ is a right (and left) approximate identity in $\mathcal{B}_2(\mathcal{H})$. This example also provides the link between Theorem 2 and its generalization, i.e. Theorem 3 below.

We will now define a positive selfadjoint operator in the Hilbert space $\mathcal{B}_2(\mathcal{H})$ induced by the Duflo-Moore operator $\hat{D}_U$. In the dense linear span $\mathsf{FR}^{\langle|}(\mathcal{H};U) \subset \mathcal{B}_2(\mathcal{H})$ — see (24) — we can define the linear operator $\mathfrak{X}_U^\circ$ as follows:

$$\mathfrak{X}_U^\circ \hat{F} := \sum_{k=1}^{\mathsf{N}} |\phi_k\rangle\langle \hat{D}_U^{-1}\psi_k|, \quad \hat{F} \in \mathsf{FR}^{\langle|}(\mathcal{H};U), \tag{154}$$

where a canonical decomposition of $\hat{F}$ is given by (26). It is easy to check that $\mathfrak{X}_U^\circ$ is a symmetric operator in the Hilbert space $\mathcal{B}_2(\mathcal{H})$; hence, it is closable. Denoting by $\mathfrak{X}_U$ the closure of $\mathfrak{X}_U^\circ$ — i.e. $\mathfrak{X}_U \equiv \overline{\mathfrak{X}_U^\circ}$ — for every complex conjugation J in $\mathcal{H}$, we have:

$$\mathfrak{X}_U = \mathfrak{U}_J\big(I \otimes (J\,\hat{D}_U^{-1}J)\big)\mathfrak{U}_J^*, \tag{155}$$

where $\mathfrak{U}_J$ is the unitary operator determined by (101). $\mathfrak{X}_U$ is a positive selfadjoint operator (since it is unitarily equivalent to $I \otimes (J\,\hat{D}_U^{-1}J)$ and $\mathrm{sp}\big(I \otimes (J\,\hat{D}_U^{-1}J)\big) = \mathrm{sp}\big(J\,\hat{D}_U^{-1}J\big) = \mathrm{sp}\big(\hat{D}_U^{-1}\big)$).

It is worth introducing the following dense linear span $\mathsf{FR}^{\langle||}(\mathcal{H};U)$ in $\mathcal{B}_2(\mathcal{H})$ (compare with definition (24)) defined by

$$\mathsf{FR}^{\langle||}(\mathcal{H};U) := \big\{\hat{F} \in \mathsf{FR}(\mathcal{H}) :\ \mathrm{Ran}(\hat{F}^*) \subset \mathrm{Dom}\big(\hat{D}_U^{-2}\big)\big\} \subset \mathsf{FR}^{\langle|}(\mathcal{H};U). \tag{156}$$

The elements of $\mathsf{FR}^{\langle||}(\mathcal{H};U)$ are those finite rank operators in $\mathcal{H}$ that admit a *canonical decomposition* of the form

$$\hat{F} = \sum_{k=1}^{\mathsf{N}} |\phi_k\rangle\langle\chi_k|, \quad \mathsf{N} \in \mathbb{N}, \tag{157}$$

where $\{\phi_k\}_{k=1}^{\mathsf{N}}$, $\{\chi_k\}_{k=1}^{\mathsf{N}}$ are linearly independent systems in $\mathcal{H}$, with $\{\chi_k\}_{k=1}^{\mathsf{N}} \subset \mathrm{Dom}\big(\hat{D}_U^{-2}\big)$. It is clear that, if $\hat{F}$ is positive selfadjoint, then one can set $\phi_k = \chi_k$ in (157) (in particular, one can always choose the vectors $\{\chi_k\}_{k=1}^{\mathsf{N}}$ mutually orthogonal). Also note that, for every $\hat{F} \in \mathsf{FR}^{\langle||}(\mathcal{H};U)$, since the operator $\mathfrak{X}_U^\circ\hat{F}$ belongs to $\mathsf{FR}^{\langle|}(\mathcal{H};U)$, we have:

$$\mathfrak{X}_U^2\hat{F} = \mathfrak{X}_U(\mathfrak{X}_U\hat{F}) = \mathfrak{X}_U(\mathfrak{X}_U^\circ\hat{F}) = \mathfrak{X}_U^\circ(\mathfrak{X}_U^\circ\hat{F}) = \sum_{k=1}^{\mathsf{N}} |\phi_k\rangle\langle\hat{D}_U^{-2}\chi_k|, \tag{158}$$

where a canonical decomposition of $\hat{F}$ is given by (157). Moreover — since, for any complex conjugation J in $\mathcal{H}$, $\mathsf{FR}^{\langle||}(\mathcal{H};U) = \mathfrak{U}_J\big(\mathcal{H} \otimes \mathrm{Dom}\big(J\,\hat{D}_U^{-2}J\big)\big)$ and $\mathfrak{X}_U^2 = \mathfrak{U}_J\big(I \otimes \big(J\,\hat{D}_U^{-2}J\big)\big)\mathfrak{U}_J^*$ — the linear span $\mathsf{FR}^{\langle||}(\mathcal{H};U)$ is a core for the positive selfadjoint operator$\mathfrak{X}_U^2$ (recall, indeed, that $\mathcal{H} \otimes \mathrm{Dom}\big(J\,\hat{D}_U^{-2}J\big)$ is a core for $I \otimes \big(J\,\hat{D}_U^{-2}J\big)$).

Let $\hat{K}$ be a bounded operator in $\mathcal{H}$ and $\hat{T}$ an Hilbert-Schmidt operator contained in the dense linear span $\mathrm{Dom}\big(\mathfrak{X}_U^2\big)$. In association with these operators, we can define the integral kernel $\gamma_U(\hat{K},\hat{T};\cdot,\cdot)\colon G \times G \to \mathbb{C}$ as follows:

$$\gamma_U(\hat{K},\hat{T};g,h) := \Big(\mathfrak{S}_U\big(\hat{K}^*\,U(g)\,(\mathfrak{X}_U^2\hat{T})^*\big)\Big)(h)^*. \tag{159}$$

Observe that — since, by virtue of the intertwining relation (49), $\big(\mathfrak{S}_U\hat{A}\big)^* = \big(\mathsf{J}_{\mathtt{m}}\,\mathfrak{S}_U(\hat{A}^*)\big)^*$, $\forall\,\hat{A} \in \mathcal{B}_2(\mathcal{H})$ — we have:

$$\gamma_U(\hat{K},\hat{T};g,h) = \mathtt{m}\,(h,h^{-1})^*\Delta_G(h)^{-\frac{1}{2}}\,\mathfrak{S}_U\big((\mathfrak{X}_U^2\hat{T})\,U(g)^*\,\hat{K}\big)(h^{-1}). \tag{160}$$

In the case where $\hat{T} \equiv \hat{F}$ is a positive selfadjoint belonging to $\mathsf{FR}^{\langle\|}(\mathcal{H};U)$ — let $\sum_{k=1}^{\mathsf{N}} |\chi_k\rangle\langle\chi_k|$ be a canonical decomposition of $\hat{F}$ — the integral kernel (159) has the following form (compare with (103)):

$$\gamma_U(\hat{K},\hat{F};g,h) = \sum_{k=1}^{\mathsf{N}} \big\langle U(g)\,\hat{D}_U^{-2}\chi_k, \hat{K}\,U(h)\,\hat{D}_U^{-1}\chi_k\big\rangle. \tag{161}$$

The following result shows that with every *suitable* right approximate identity in $\mathcal{B}_2(\mathcal{H})$ is associated a formula for the $\hat{K}$-deformed star product $(\cdot) \overset{U}{\underset{\hat{K}}{\star}} (\cdot)$ in $\mathrm{L}^2(G)$, formula that (taking into account (161)) extends the first of the expressions (131) for the star product.

Theorem 3 *Let $\hat{K}$ be a bounded operator in $\mathcal{H}$, and let $\{\hat{T}_n\}_{n\in\mathbb{N}}$ be a right approximate identity in the* H**-algebra $\mathcal{B}_2(\mathcal{H})$ such that $\{\hat{T}_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}\big(\mathfrak{X}_U^2\big)$. Then, for any $f_1, f_2 \in \mathrm{L}^2(G)$, the following formula holds:*

$$\begin{aligned} f_1 \overset{U}{\underset{\hat{K}}{\star}} f_2 = {}_{\|\cdot\|_{\mathrm{L}^2}}\lim_{n\to\infty} \int_G \mathrm{d}\mu_G(h)\; f_1(h)\,\mathtt{m}\,(h,h^{-1}(\cdot))^*\Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \\ \times \int_G \mathrm{d}\mu_G(h')\;\gamma_U(\hat{K},\hat{T}_n;h^{-1}(\cdot),h')\,f_2(h'). \end{aligned} \tag{162}$$

Proof: Since the proof goes along lines similar to those already traced in Subsects. 5.2 and 5.3, we will be rather sketchy. For the sake of clarity, we subdivide our argument into a few steps.

1) Let us first show that, for every $\hat{F} \in \mathsf{FR}^{\langle\|}(\mathcal{H};U) \subset \mathrm{Dom}\big(\mathfrak{X}_U^2\big)$ and every $f \in \mathrm{L}^2(G)$, we have:

$$\int_G \mathrm{d}\mu_G(h)\;\gamma_U(\hat{K},\hat{F};\cdot,h)\,f(h) = \mathfrak{S}_U\big(\hat{K}(\mathfrak{S}_U^* f)(\mathfrak{X}_U\hat{F})\big). \tag{163}$$

In fact, if $\sum_{k=1}^{\mathsf{N}} |\phi_k\rangle\langle\chi_k|$ is a canonical decomposition of $\hat{F}$, taking into account relation (158) we find that

$$\begin{aligned} \int_G \mathrm{d}\mu_G(h)\;\gamma_U(\hat{K},\hat{F};\cdot,h)\,f(h) &= \big\langle \mathfrak{S}_U\big(\hat{K}^*\,U(\cdot)\,(\mathfrak{X}_U^2\hat{F})^*\big), f\big\rangle_{\mathrm{L}^2} \\ &= \big\langle \hat{K}^*\,U(\cdot)\,(\mathfrak{X}_U^2\hat{F})^*, \mathfrak{S}_U^* f\big\rangle_{\mathcal{B}_2} \\ &= \sum_{k=1}^{\mathsf{N}} \mathsf{tr}\big(|\phi_k\rangle\langle\hat{D}_U^{-2}\chi_k|\,U(\cdot)^*\,\hat{K}\,(\mathfrak{S}_U^* f)\big) \\ &= \sum_{k=1}^{\mathsf{N}} \mathfrak{S}_U\big(\hat{K}\,(\mathfrak{S}_U^* f)\,|\phi_k\rangle\langle\hat{D}_U^{-1}\chi_k|\big) \\ &= \mathfrak{S}_U\big(\hat{K}(\mathfrak{S}_U^* f)(\mathfrak{X}_U\hat{F})\big). \end{aligned} \tag{164}$$

2) Let us observe the following fact. Let $\hat{C}$ be a selfadjoint operator in a (complex separable) Hilbert space $\mathcal{S}$, and let $\mathcal{S}_0$ be a dense linear span in $\mathcal{S}$ contained in $\mathrm{Dom}\big(\hat{C}^2\big)$; we suppose, furthermore, that $\mathcal{S}_0$ is a core for the positive selfadjoint operator $\hat{C}^2$. Then, for every vector $\varphi \in \mathrm{Dom}\big(\hat{C}^2\big)$, there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_0$ such that both $\lim_{n\to\infty}\|\varphi-\varphi_n\| = 0$ and $\lim_{n\to\infty}\|\hat{C}^2(\varphi-\varphi_n)\| = 0$. These two relations imply that $\lim_{n\to\infty}\|\hat{C}(\varphi-\varphi_n)\| = 0$, as well. Indeed, this fact can be easily checked in the case where $\mathcal{S}$ is a space of square integrable functions and $\hat{C}$ is a multiplication operator by a measurable function — hint: an unbounded multiplication operator can be written as the sum of a (bounded) multiplication operator by a function of modulus not larger than

one and, possibly, of a multiplication operator by a function of modulus larger than one, these two functions having disjoint supports. Hence, by virtue of the spectral theorem in ‘multiplication operator form’ — see [30] — this property holds true for a generic selfadjoint operator $\hat{C}$.

3) It is possible to generalize formula (163); i.e. one can prove that, for every $\hat{T} \in \mathrm{Dom}\big(\mathfrak{X}_U^2\big)$ and every $f \in \mathrm{L}^2(G)$, the following relation holds:

$$\int_G \mathrm{d}\mu_G(h)\; \gamma_U(\hat{K}, \hat{T}; \cdot, h)\, f(h) = \mathfrak{S}_U\big(\hat{K}(\mathfrak{S}_U^* f)(\mathfrak{X}_U \hat{T})\big). \tag{165}$$

Indeed, let $\{\hat{F}_l\}_{l\in\mathbb{N}}$ be a sequence in $\mathsf{FR}^{\langle\mathbb{I}}(\mathcal{H};U)$ (which is a core for $\mathfrak{X}_U^2$) such that ${}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{l\to\infty} \hat{F}_l = \hat{T}$ and ${}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{l\to\infty} \mathfrak{X}_U^2 \hat{F}_l = \mathfrak{X}_U^2 \hat{T}$. Then, we know that ${}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{l\to\infty} \mathfrak{X}_U \hat{F}_l = \mathfrak{X}_U \hat{T}$, as well. Observe now that by relation (163) we have:

$$\begin{aligned}
{}_{\|\cdot\|_{\mathrm{L}^2}}\lim_{l\to\infty} \int_G \mathrm{d}\mu_G(h)\; \gamma_U(\hat{K}, \hat{F}_l; \cdot, h)\, f(h) &= {}_{\|\cdot\|_{\mathrm{L}^2}}\lim_{l\to\infty} \mathfrak{S}_U\big(\hat{K}(\mathfrak{S}_U^* f)(\mathfrak{X}_U \hat{F}_l)\big) \\
&= \mathfrak{S}_U\Big(\hat{K}(\mathfrak{S}_U^* f)\; {}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{l\to\infty} (\mathfrak{X}_U \hat{F}_l)\Big) \\
&= \mathfrak{S}_U\big(\hat{K}(\mathfrak{S}_U^* f)(\mathfrak{X}_U \hat{T})\big), \quad \forall f \in \mathrm{L}^2(G).
\end{aligned} \tag{166}$$

On the other hand, for every $f \in \mathrm{L}^2(G)$ and every $g \in G$, we also have that

$$\begin{aligned}
\lim_{l\to\infty} \int_G \mathrm{d}\mu_G(h)\; \gamma_U(\hat{K}, \hat{F}_l; g, h)\, f(h) &= \lim_{l\to\infty} \big\langle \mathfrak{S}_U\big(\hat{K}^* U(g)\, (\mathfrak{X}_U^2 \hat{F}_l)^*\big), f\big\rangle_{\mathrm{L}^2} \\
&= \Big\langle \mathfrak{S}_U\big(\hat{K}^* U(g)\; {}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{l\to\infty} (\mathfrak{X}_U^2 \hat{F}_l)^*\big), f\Big\rangle_{\mathrm{L}^2} \\
&= \big\langle \mathfrak{S}_U\big(\hat{K}^* U(g)\, (\mathfrak{X}_U^2 \hat{T})^*\big), f\big\rangle_{\mathrm{L}^2} \\
&= \int_G \mathrm{d}\mu_G(h)\; \gamma_U(\hat{K}, \hat{T}; g, h)\, f(h).
\end{aligned} \tag{167}$$

From (166) and (167) it follows that relation (165) holds true.

4) Let us now prove that, for every $\hat{T} \in \mathrm{Dom}\big(\mathfrak{X}_U^2\big)$ and any pair of functions $f_1, f_2 \in \mathrm{L}^2(G)$,

$$\begin{aligned}
&\int_G \mathrm{d}\mu_G(h)\; f_1(h)\, \mathtt{m}\,(h, h^{-1}(\cdot))^* \Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \int_G \mathrm{d}\mu_G(h')\; \gamma_U(\hat{K}, \hat{T}; h^{-1}(\cdot), h')\, f_2(h') \\
&= \langle U(\cdot)\, (\mathfrak{X}_U \hat{T})^* (\mathfrak{S}_U^* f_2)^* \hat{K}^*, \mathfrak{S}_U^* f_1\rangle_{\mathcal{B}_2}.
\end{aligned} \tag{168}$$

Indeed, according to relations (165) and (112), for any $f_1, f_2 \in \mathrm{L}^2(G)$ we have:

$$\begin{aligned}
&\int_G \mathrm{d}\mu_G(h)\; f_1(h)\, \mathtt{m}\,(h, h^{-1}(\cdot))^* \Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \int_G \mathrm{d}\mu_G(h')\; \gamma_U(\hat{K}, \hat{T}; h^{-1}(\cdot), h')\, f_2(h') \\
&= \int_G \mathrm{d}\mu_G(h)\; f_1(h)\, \mathtt{m}\,(h, h^{-1}(\cdot))^* \Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \Big(\mathfrak{S}_U\big(\hat{K}(\mathfrak{S}_U^* f_2)(\mathfrak{X}_U \hat{T})\big)\Big)(h^{-1}(\cdot)) \\
&= \big\langle R_{\mathtt{m}}(\cdot)\, \mathsf{J}_{\mathtt{m}}\, \mathfrak{S}_U\big(\hat{K}(\mathfrak{S}_U^* f_2)(\mathfrak{X}_U \hat{T})\big), f_1\big\rangle_{\mathrm{L}^2}.
\end{aligned} \tag{169}$$

Then, using the intertwining relation (49) and relation (46), we find that

$$\begin{aligned}
&\int_G \mathrm{d}\mu_G(h)\; f_1(h)\, \mathtt{m}\,(h, h^{-1}(\cdot))^* \Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \int_G \mathrm{d}\mu_G(h')\; \gamma_U(\hat{K}, \hat{T}; h^{-1}(\cdot), h')\, f_2(h') \\
&= \big\langle R_{\mathtt{m}}(\cdot)\, \mathfrak{S}_U\big((\mathfrak{X}_U \hat{T})^* (\mathfrak{S}_U^* f_2)^* \hat{K}^*\big), f_1\big\rangle_{\mathrm{L}^2} \\
&= \big\langle \mathfrak{S}_U\big(U(\cdot)\, (\mathfrak{X}_U \hat{T})^* (\mathfrak{S}_U^* f_2)^* \hat{K}^*\big), f_1\big\rangle_{\mathrm{L}^2} \\
&= \langle U(\cdot)\, (\mathfrak{X}_U \hat{T})^* (\mathfrak{S}_U^* f_2)^* \hat{K}^*, \mathfrak{S}_U^* f_1\rangle_{\mathcal{B}_2}.
\end{aligned} \tag{170}$$

5) The next step is to show that, for every $\hat{T} \in \mathrm{Dom}\big(\mathfrak{X}_U^2\big)$ and any $f_1, f_2 \in \mathrm{L}^2(G)$,

$$\langle U(\cdot)\, (\mathfrak{X}_U \hat{T})^* (\mathfrak{S}_U^* f_2)^* \hat{K}^*, \mathfrak{S}_U^* f_1 \rangle_{\mathcal{B}_2} = \mathfrak{S}_U \, \mathfrak{R}_{\hat{T}}\big((\mathfrak{S}_U^* f_1)\, \hat{K}\, (\mathfrak{S}_U^* f_2)\big). \tag{171}$$

We will first give a proof of this relation in the case where $\hat{T} \equiv \hat{F} \in \mathsf{FR}^{\langle\|}(\mathcal{H}; U)$. In fact, considering a canonical decomposition $\sum_{k=1}^{\mathsf{N}} |\phi_k\rangle\langle\chi_k|$ of $\hat{F}$, we get:

$$\begin{aligned} \langle U(\cdot)\, (\mathfrak{X}_U \hat{F})^* (\mathfrak{S}_U^* f_2)^* \hat{K}^*, \mathfrak{S}_U^* f_1 \rangle_{\mathcal{B}_2} &= \sum_{k=1}^{\mathsf{N}} \big\langle |U(\cdot)\, \hat{D}_U^{-1} \chi_k\rangle\langle \hat{K}\, (\mathfrak{S}_U^* f_2)\, \phi_k|, \mathfrak{S}_U^* f_1 \big\rangle_{\mathcal{B}_2} \\ &= \sum_{k=1}^{\mathsf{N}} \mathsf{tr}\big(|\hat{K}\, (\mathfrak{S}_U^* f_2)\, \phi_k\rangle\, \langle U(\cdot)\, \hat{D}_U^{-1} \chi_k|\, \mathfrak{S}_U^* f_1\big) \\ &= \sum_{k=1}^{\mathsf{N}} \mathfrak{S}_U\big((\mathfrak{S}_U^* f_1)\, \hat{K}\, (\mathfrak{S}_U^* f_2)\, |\phi_k\rangle\langle\chi_k|\big) \\ &= \mathfrak{S}_U\big((\mathfrak{S}_U^* f_1)\, \hat{K}\, (\mathfrak{S}_U^* f_2)\, \hat{F}\big). \end{aligned} \tag{172}$$

Next, let $\hat{T}$ be a generic Hilbert-Schmidt operator in $\mathrm{Dom}\big(\mathfrak{X}_U^2\big)$, and let $\{\hat{F}_l\}_{l\in\mathbb{N}}$ be a sequence in $\mathsf{FR}^{\langle\|}(\mathcal{H}; U)$ such that ${}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{l\to\infty} \hat{F}_l = \hat{T}$ and ${}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{l\to\infty} \mathfrak{X}_U^2 \hat{F}_l = \mathfrak{X}_U^2 \hat{T}$ (hence: ${}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{l\to\infty} \mathfrak{X}_U \hat{F}_l = \mathfrak{X}_U \hat{T}$). Then, for any $f_1, f_2 \in \mathrm{L}^2(G)$, we have:

$$\begin{aligned} {}_{\|\cdot\|_{\mathrm{L}^2}}\lim_{l\to\infty} \langle U(\cdot)\, (\mathfrak{X}_U \hat{F}_l)^* (\mathfrak{S}_U^* f_2)^* \hat{K}^*, \mathfrak{S}_U^* f_1 \rangle_{\mathcal{B}_2} &= {}_{\|\cdot\|_{\mathrm{L}^2}}\lim_{l\to\infty} \mathfrak{S}_U \, \mathfrak{R}_{\hat{F}_l}\big((\mathfrak{S}_U^* f_1)\, \hat{K}\, (\mathfrak{S}_U^* f_2)\big) \\ &= \mathfrak{S}_U \, \mathfrak{R}_{\hat{T}}\big((\mathfrak{S}_U^* f_1)\, \hat{K}\, (\mathfrak{S}_U^* f_2)\big), \end{aligned} \tag{173}$$

where we have used relation (172) and the fact that ${}_{\|\cdot\|_{\mathcal{B}_2}}\lim_{l\to\infty} \mathfrak{R}_{\hat{F}_l} \hat{A} = \mathfrak{R}_{\hat{T}} \hat{A}$, for every $\hat{A} \in \mathcal{B}_2(\mathcal{H})$. Moreover, we also have that

$$\lim_{l\to\infty} \langle U(g)\, (\mathfrak{X}_U \hat{F}_l)^* (\mathfrak{S}_U^* f_2)^* \hat{K}^*, \mathfrak{S}_U^* f_1 \rangle_{\mathcal{B}_2} = \langle U(g)\, (\mathfrak{X}_U \hat{T})^* (\mathfrak{S}_U^* f_2)^* \hat{K}^*, \mathfrak{S}_U^* f_1 \rangle_{\mathcal{B}_2}, \tag{174}$$

for all $g \in G$. From (173) and (174) it follows eventually that relation (168) holds true.

6) Relations (168) and (171) imply that, for every $\hat{T} \in \mathrm{Dom}\big(\mathfrak{X}_U^2\big)$ and any pair of functions $f_1, f_2 \in \mathrm{L}^2(G)$,

$$\begin{aligned} &\int_G \mathrm{d}\mu_G(h)\; f_1(h)\, \mathtt{m}\,(h, h^{-1}(\cdot))^* \Delta_G(h^{-1}(\cdot))^{\frac{1}{2}} \int_G \mathrm{d}\mu_G(h')\; \gamma_U(\hat{K}, \hat{T}; h^{-1}(\cdot), h')\, f_2(h') \\ &= \mathfrak{S}_U \, \mathfrak{R}_{\hat{T}}\big((\mathfrak{S}_U^* f_1)\, \hat{K}\, (\mathfrak{S}_U^* f_2)\big). \end{aligned} \tag{175}$$

7) Finally, as in the proof of Theorem 2, one can exploit relation (175), the boundedness of the Wigner map and the fact that the sequence $\{\hat{T}_n\}_{n\in\mathbb{N}}$ is a right approximate identity for obtaining formula (162).

The proof is complete. □

6 Applications

In this section, we will consider two simple — but extremely significant — applications of the theory developed in Sects. 3-5. We will first consider the case of a square integrable —

genuinely projective — representation of a unimodular group; i.e., the group of translations on phase space. The analysis of this case leads to the Grönewold-Moyal star product, i.e. the prototype of star product. Next, we will study a case where square integrable unitary representations are involved of a group which is *not* unimodular; namely, the 1-dimensional affine group. As already mentioned, this group is at the base of wavelet analysis.

6.1 The group of translations on phase space

Let us consider the group of translations on the $(1+1)$-dimensional phase space, namely, the additive group $\mathbb{R}\times\mathbb{R}$ (the extension to the $(n+n)$-dimensional case is straightforward). As is well known (see, e.g., ref. [34]), the map

$$\mathbb{R}\times\mathbb{R}\ni(q,p)\mapsto U(q,p)\in\mathcal{U}(\mathrm{L}^2(\mathbb{R})), \tag{176}$$

defined by

$$\begin{aligned} U(q,p) &:= \exp(\mathrm{i}\,(p\,\hat{q}-q\hat{p})) \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2}qp}\,\exp(\mathrm{i}p\,\hat{q})\,\exp(-\mathrm{i}q\hat{p}) = \mathrm{e}^{\frac{\mathrm{i}}{2}qp}\,\exp(-\mathrm{i}q\hat{p})\,\exp(\mathrm{i}p\,\hat{q}),\quad q,p\in\mathbb{R}, \end{aligned} \tag{177}$$

— where $\hat{q}$, $\hat{p}$ are the standard position and momentum operators — is a projective representation of the unimodular group $\mathbb{R}\times\mathbb{R}$, representation which we will call (with a slight abuse of terminology) *Weyl system*. The Weyl system is — as already observed in Sect. 2 — a square integrable representation. It 'encodes' the canonical commutation relations of quantum mechanics (in the integrated form), as shown by the last two members of (177).

The (generalized) Wigner transform associated with the Weyl system is not the standard Wigner transform but the so-called *Fourier-Wigner transform* [35]. In fact, it turns out that these maps are related by the *symplectic Fourier transform*, i.e. by the unitary operator $\mathcal{F}_{\mathrm{sp}}\colon\mathrm{L}^2(\mathbb{R}\times\mathbb{R})\rightarrow\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ determined by

$$\big(\mathcal{F}_{\mathrm{sp}}f\big)(q,p)=\frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}}f(q',p')\,\mathrm{e}^{\mathrm{i}\,(qp'-pq')}\,\mathrm{d}q'\mathrm{d}p',\quad\forall\,f\in\mathrm{L}^1(\mathbb{R}\times\mathbb{R})\cap\mathrm{L}^2(\mathbb{R}\times\mathbb{R}). \tag{178}$$

Recall that $\mathcal{F}_{\mathrm{sp}}$ enjoys the remarkable property of being both unitary and selfadjoint:

$$\mathcal{F}_{\mathrm{sp}}=\mathcal{F}_{\mathrm{sp}}^{*},\quad \mathcal{F}_{\mathrm{sp}}^{2}=I. \tag{179}$$

As already mentioned in Sect. 2, $(2\pi)^{-1}\mathrm{d}q\mathrm{d}p$ is the Haar measure on $\mathbb{R}\times\mathbb{R}$ normalized in agreement with the Weyl system U. Then, in this case, the generalized Wigner transform $\mathfrak{S}_U$ is the isometry from $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R}))$ into $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})\equiv\mathrm{L}^2\big(\mathbb{R}\times\mathbb{R},(2\pi)^{-1}\mathrm{d}q\mathrm{d}p;\mathbb{C}\big)$ determined by

$$\big(\mathfrak{S}_U\,\hat{\rho}\big)(q,p)=\mathtt{tr}(U(q,p)^{*}\hat{\rho}),\quad\forall\,\hat{\rho}\in\mathcal{B}_1(\mathrm{L}^2(\mathbb{R})). \tag{180}$$

The multiplier $\mathtt{m}:(\mathbb{R}\times\mathbb{R})\times(\mathbb{R}\times\mathbb{R})\rightarrow\mathbb{T}$ associated with U is of the form

$$\mathtt{m}\,(q,p\,;q',p')=\exp\Big(\frac{\mathrm{i}}{2}(qp'-pq')\Big). \tag{181}$$

Hence, for the function $\overleftrightarrow{\mathtt{m}}:(\mathbb{R}\times\mathbb{R})\times(\mathbb{R}\times\mathbb{R})\rightarrow\mathbb{T}$ (see (41)) we find, in this case, the following expression:

$$\overleftrightarrow{\mathtt{m}}(q,p\,;q',p')=\mathtt{m}\,(q,p\,;q'-q,p'-p)^{*}\,\mathtt{m}\,(q'-q,p'-p\,;q,p)=\exp\big(-\,\mathrm{i}\,(qp'-pq')\big). \tag{182}$$

Therefore, according to formula (40), we have that the generalized Wigner transform $\mathfrak{S}_U$ intertwines the unitary representation

$$U \vee U \colon \mathbb{R} \times \mathbb{R} \to \mathcal{U}(\mathcal{B}_2(\mathrm{L}^2(\mathbb{R}))) \tag{183}$$

with the representation $\mathcal{T}_{\mathtt{m}} : \mathbb{R} \times \mathbb{R} \to \mathcal{U}(\mathrm{L}^2(\mathbb{R} \times \mathbb{R}))$ defined by

$$\big(\mathcal{T}_{\mathtt{m}}(q,p)\, f\big)(q',p') = \mathrm{e}^{-\mathrm{i}\,(qp'-pq')}\, f(q',p'), \quad \forall\, f \in \mathrm{L}^2(\mathbb{R} \times \mathbb{R}). \tag{184}$$

Moreover, $\mathfrak{S}_U$ intertwines the involution $\mathfrak{J}$ in $\mathcal{B}_2(\mathcal{H})$ with the complex conjugation $\mathsf{J} \equiv \mathsf{J}_{\mathtt{m}}$ that, in this case — as the reader may readily check — takes the following form:

$$\big(\mathsf{J} f\big)(q,p) = f(-q,-p)^*, \quad \forall\, f \in \mathrm{L}^2(\mathbb{R} \times \mathbb{R}). \tag{185}$$

As anticipated, the *standard Wigner transform* — we will denote it by $\mathfrak{T}$ — is the isometry obtained composing the isometry $\mathfrak{S}_U$, determined by (180), with the symplectic Fourier transform (see [17]):

$$\mathfrak{T} := \mathcal{F}_{\mathrm{sp}}\, \mathfrak{S}_U \colon \mathcal{B}_2(\mathrm{L}^2(\mathbb{R})) \to \mathrm{L}^2(\mathbb{R} \times \mathbb{R}). \tag{186}$$

It is clear that the isometry $\mathfrak{T}$ intertwines the representation $U \vee U$ with the unitary representation $\mathcal{V} \colon \mathbb{R} \times \mathbb{R} \to \mathcal{U}(\mathrm{L}^2(\mathbb{R} \times \mathbb{R}))$ defined by

$$\mathcal{V}(q,p) := \mathcal{F}_{\mathrm{sp}}\, \mathcal{T}_{\mathtt{m}}(q,p)\, \mathcal{F}_{\mathrm{sp}}, \quad \forall\, (q,p) \in \mathbb{R} \times \mathbb{R}; \tag{187}$$

as the reader may easily check, explicitly, we have:

$$\big(\mathcal{V}(q,p)\, f\big)(q',p') = f(q'-q, p'-p), \quad \forall\, f \in \mathrm{L}^2(\mathbb{R} \times \mathbb{R}). \tag{188}$$

Thus, the representation $\mathcal{V}$ acts by simply translating functions on phase space. It is also a remarkable fact — see ref. [36] — that

$$\mathrm{Ran}(\mathfrak{T}) = \mathrm{L}^2(\mathbb{R} \times \mathbb{R}); \tag{189}$$

equivalently, $\mathcal{R}_U \equiv \mathrm{Ran}\big(\mathfrak{S}_U\big) = \mathrm{L}^2(\mathbb{R} \times \mathbb{R})$ (this fact can be verified deducing the integral kernel of the Hilbert-Schmidt operator $\mathfrak{S}_U^* f$, for a generic $f \in \mathrm{L}^2(\mathbb{R} \times \mathbb{R})$, and observing that $\mathrm{Ker}\big(\mathfrak{S}_U^*\big) = \{0\}$). Therefore, the standard Wigner transform $\mathfrak{T}$ — and its adjoint $\mathfrak{T}^*$, the *standard Weyl map* — are both unitary operators.

Let us now study the star product in $\mathrm{L}^2(\mathbb{R} \times \mathbb{R})$ induced by the Weyl system U. Recalling Theorem 1, and taking into account the fact that, in this case, $\mathcal{R}_U = \mathrm{L}^2(\mathbb{R} \times \mathbb{R})$ (and $d_U = 1$), we have:

$$\begin{aligned}\Big(f_1 \overset{U}{\star} f_2\Big)(q,p) &= \frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{R}} f_1(q',p')\, \mathtt{m}\,(q,p\,; q-q', p-p')^*\, f_2(q-q', p-p')\, \mathrm{d}q'\mathrm{d}p' \\ &= \frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{R}} f_1(q',p')\, f_2(q-q', p-p')\, \exp\Big(\frac{\mathrm{i}}{2}(qp'-pq')\Big)\, \mathrm{d}q'\mathrm{d}p',\end{aligned} \tag{190}$$

$\forall\, f_1, f_2 \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R})$. Thus, the star product associated with the Weyl system is nothing but the *twisted convolution* of functions [20, 35]. According to the results of Sect. 4, $\big(\mathrm{L}^2(\mathbb{R} \times \mathbb{R}), \overset{U}{\star}, \mathsf{J}\big)$ is a proper H*-algebra and $\mathfrak{S}_U \colon \mathcal{B}_2(\mathcal{H}) \to \mathrm{L}^2(\mathbb{R} \times \mathbb{R})$ is an isomorphism of H*-algebras.

The unitary operators $\mathfrak{T}$, $\mathfrak{T}^*$ induce another star product of functions

$$(\cdot) \circledast (\cdot) \colon \mathrm{L}^2(\mathbb{R} \times \mathbb{R}) \times \mathrm{L}^2(\mathbb{R} \times \mathbb{R}) \ni (f_1, f_2) \mapsto \mathfrak{T}\Big(\big(\mathfrak{T}^* f_1\big)\big(\mathfrak{T}^* f_2\big)\Big) \in \mathrm{L}^2(\mathbb{R} \times \mathbb{R}), \tag{191}$$

namely, the *twisted product* (see [20]). Using the fact that $\mathfrak{T} = \mathcal{F}_{\rm sp}\, \mathfrak{S}_U$ and $\mathfrak{T}^* = \mathfrak{S}_U\, \mathcal{F}_{\rm sp}$, we obtain that

$$f_1 \circledast f_2 = \mathcal{F}_{\rm sp}\Big(\big(\mathcal{F}_{\rm sp} f_1\big) \overset{U}{\star} \big(\mathcal{F}_{\rm sp} f_2\big)\Big). \tag{192}$$

From this relation, by an explicit calculation, one finds that, for any pair of functions f_1, f_2 in $\mathrm{L}^1(\mathbb{R}\times\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R}\times\mathbb{R})$,

$$\big(f_1 \circledast f_2\big)(q,p) = \frac{1}{\pi^2}\int_{\mathbb{R}\times\mathbb{R}} \mathrm{d}q'\mathrm{d}p' \int_{\mathbb{R}\times\mathbb{R}} \mathrm{d}q''\mathrm{d}p''\; \theta\big(q,p;q',p';q'',p''\big)\, f_1(q',p')\, f_2(q'',p''), \tag{193}$$

where we have set:

$$\theta\big(q,p;q',p';q'',p''\big) := \exp\big(\mathrm{i}2(qp' - pq' + q'p'' - p'q'' + q''p - p''q)\big). \tag{194}$$

The function $\theta\colon (\mathbb{R}\times\mathbb{R})\times(\mathbb{R}\times\mathbb{R})\times(\mathbb{R}\times\mathbb{R}) \to \mathbb{T}$ is the celebrated *Grönewold-Moyal kernel*. The symplectic Fourier transform intertwines the complex conjugation J with the standard complex conjugation in $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$: $\mathcal{F}_{\rm sp}\,\mathsf{J}\,\mathcal{F}_{\rm sp} f = f^*$. Therefore, $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ endowed with the twisted product and with the standard complex conjugation is again a proper H*-algebra. To the best of our knowledge, this fact has been noted for the first time by Pool [36].

6.2 The 1-dimensional affine group

Let us consider, now, the 1-*dimensional affine group*, namely, the semi-direct product group $G = \mathbb{R} \rtimes \mathbb{R}_*^+$, where $\mathbb{R}_*^+$ is the subgroup of *dilations*; i.e. $\mathbb{R}_*^+$ is the group of strictly positive real numbers (we will denote by $\mathbb{R}_*^-$ the set of strictly negative real numbers) which acts multiplicatively on $\mathbb{R}$. Thus, G consists of the topological space $\mathbb{R}\times\mathbb{R}_*^+$, endowed with the composition law

$$(a,r)\,(a',r') = (a + ra', rr'), \quad a\in\mathbb{R},\ r\in\mathbb{R}_*^+. \tag{195}$$

This group is not unimodular. A pair $\mu_{\mathtt{L}}$, $\mu_{\mathtt{R}}$ of — left and right, respectively — conjugated Haar measures on G ($\int_G f(g)\,\mathrm{d}\mu_{\mathtt{L}}(g) = \int_G f(g^{-1})\,\mathrm{d}\mu_{\mathtt{R}}(g)$) are given by

$$\mathrm{d}\mu_{\mathtt{L}}(a,r) = r^{-2}\,\mathrm{d}a\,\mathrm{d}r, \quad \mathrm{d}\mu_{\mathtt{R}}(a,r) = r^{-1}\,\mathrm{d}a\,\mathrm{d}r, \quad a\in\mathbb{R},\ r\in\mathbb{R}_*^+. \tag{196}$$

Hence, the modular function Δ_G on G is given by $\Delta_G(a,r) = r^{-1}$, $\forall a\in\mathbb{R}$, $\forall r\in\mathbb{R}_*^+$. As already recalled in Sect. 2, this group is at the base of the theory of the wavelet transform. For the sake of completeness, we will come back to this point later on. It is also worth mentioning that the quantization-dequantization theory based on the affine group has been studied by Aslaksen and Klauder [37], who obtained the Wigner and Weyl maps associated with the representations this group. However, they did not consider the concept of star product.

Using Mackey's little group method for classifying the irreducible representations of semi-direct products with abelian normal factors (see [21]), and the results of ref. [38] on the characterization of square integrable representations of the groups of this type, one finds out that G admits a maximal set of (unitarily) inequivalent square integrable irreducible representations consisting of two elements: $\{U^{(-)}\colon G \to \mathcal{U}(\mathrm{L}^2(\mathbb{R}_*^-)), U^{(+)}\colon G\to\mathcal{U}(\mathrm{L}^2(\mathbb{R}_*^+))\}$. These two unitary representations are defined by:

$$\big(U^{(-)}(a,r)\,\varphi^{(-)}\big)(x) := r^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}ax}\varphi^{(-)}(rx), \quad a\in\mathbb{R},\ r\in\mathbb{R}_*^+,\ x\in\mathbb{R}_*^-,\ \varphi^{(-)}\in\mathrm{L}^2(\mathbb{R}_*^-), \tag{197}$$

$$\big(U^{(+)}(a,r)\,\varphi^{(+)}\big)(x) := r^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}ax}\varphi^{(+)}(rx), \quad a\in\mathbb{R},\ r\in\mathbb{R}_*^+,\ x\in\mathbb{R}_*^+,\ \varphi^{(+)}\in\mathrm{L}^2(\mathbb{R}_*^+), \tag{198}$$

where the Hilbert space $\mathrm{L}^2(\mathbb{R}_*^\pm)$ is of course defined considering the restriction to $\mathbb{R}_*^\pm$ of the Lebesgue measure on $\mathbb{R}$. Moreover, by the results of ref. [38], the Duflo-Moore operator $\hat{D}_{(\pm)}$

associated with the representation $U^{(\pm)}$ — and normalized according to $\mu_{\mathtt{L}}$ — is the unbounded multiplication operator (defined on its natural domain) by the function

$$\mathbb{R}_*^{\pm} \ni x \mapsto \left(\frac{2\pi}{|x|}\right)^{\frac{1}{2}}. \tag{199}$$

The representations $U^{(-)}$, $U^{(+)}$ are unitarily inequivalent, but they are intertwined by the aniunitary operator $\mathfrak{Z}\colon \mathrm{L}^2(\mathbb{R}_*^-) \ni \varphi \mapsto \varphi(-(\cdot))^* \in \mathrm{L}^2(\mathbb{R}_*^+)$. Hence, they are physically equivalent. We will denote by $\mathfrak{S}_{(-)}$ and $\mathfrak{S}_{(+)}$, respectively, the associated Wigner maps. These maps are isometries that intertwine the unitary representations $U^{(-)} \vee U^{(-)}$ and $U^{(+)} \vee U^{(+)}$, respectively, with the two-sided regular representation $\mathcal{T}$ of $\mathbb{R} \rtimes \mathbb{R}_*^+$, representation which is defined by

$$\big(\mathcal{T}(a,r)\, f\big)(a',r') := r^{-\frac{1}{2}}\, f\big(r^{-1}(a' - a + r'a), r'\big), \quad \forall f \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R}_*^+, \mu_{\mathtt{L}}). \tag{200}$$

The standard involutions $\mathfrak{J}_{(-)}$, $\mathfrak{J}_{(+)}$ in the Hilbert-Schmidt spaces $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R}_*^-))$, $\mathcal{B}_2(\mathrm{L}^2(\mathbb{R}_*^+))$ are intertwined by the Wigner maps $\mathfrak{S}_{(-)}$ and $\mathfrak{S}_{(+)}$, respectively, with the map

$$\mathsf{J}\colon \mathrm{L}^2(\mathbb{R}\times\mathbb{R}_*^+, \mu_{\mathtt{L}}) \to \mathrm{L}^2(\mathbb{R}\times\mathbb{R}_*^+, \mu_{\mathtt{L}}), \tag{201}$$

which is the complex conjugation defined by

$$\big(\mathsf{J} f\big)(a,r) = r^{\frac{1}{2}}\, f(-r^{-1}a, r^{-1})^*, \quad \forall f \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R}_*^+, \mu_{\mathtt{L}}). \tag{202}$$

The explicit form of the Weyl map $\mathfrak{S}_{(\pm)}^*\colon \mathrm{L}^2(G) \to \mathcal{B}_2(\mathrm{L}^2(\mathbb{R}_*^{\pm}))$ can be easily obtained applying formula (60). Indeed, for every function $\mathsf{f}\colon G \to \mathbb{C}$ in $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ and every vector $\varphi^{(\pm)}$ in $\mathrm{Dom}\big(\hat{D}_{(\pm)}^{-1}\big)$, we have:

$$\begin{aligned}\Big(\big(\mathfrak{S}_{(\pm)}^* \mathsf{f}\big)\, \varphi^{(\pm)}\Big)(x) &= \big(\hat{U}(\mathsf{f})\, \hat{D}_{(\pm)}^{-1}\, \varphi^{(\pm)}\big)(x) \\ &= \int_G \mathsf{f}(a,r)\, \big(U^{(\pm)}(a,r)\, \hat{D}_{(\pm)}^{-1} \varphi^{(\pm)}\big)(x)\, \mathrm{d}\mu_{\mathtt{L}}(a,r) \\ &= \int_G \mathsf{f}(a,r)\, \sqrt{r}\, \mathrm{e}^{\mathrm{i}\, ax} \sqrt{\frac{r|x|}{2\pi}}\, \varphi^{(\pm)}(rx)\, \mathrm{d}\mu_{\mathtt{L}}(a,r), \quad \text{for a.a. } x \in \mathbb{R}_*^{\pm}. \end{aligned} \tag{203}$$

Next, by virtue of Fubini's theorem and of a change of variables ($r \mapsto x^{-1}y$, with $x, y \in \mathbb{R}_*^{\pm}$), we get:

$$\begin{aligned}\Big(\big(\mathfrak{S}_{(\pm)}^* \mathsf{f}\big)\, \varphi^{(\pm)}\Big)(x) &= |x|^{\frac{1}{2}} \int_{\mathbb{R}_*^{\pm}} |y|^{-1} \mathrm{d}y\; \varphi^{(\pm)}(y) \int_{\mathbb{R}} \frac{\mathrm{d}a}{\sqrt{2\pi}}\, \mathsf{f}(a, x^{-1}y)\, \mathrm{e}^{\mathrm{i}\, ax} \\ &= \int_{\mathbb{R}_*^{\pm}} \varsigma_{\mathsf{f}}^{(\pm)}(x,y)\, \varphi^{(\pm)}(y)\, \mathrm{d}y, \quad \text{for a.a. } x \in \mathbb{R}_*^{\pm}, \end{aligned} \tag{204}$$

where, for every $f \in \mathrm{L}^2(G)$, the integral kernel $\varsigma_f^{(\pm)}(\cdot,\cdot)\colon \mathbb{R}_*^{\pm} \times \mathbb{R}_*^{\pm} \to \mathbb{C}$ is defined by

$$\varsigma_f^{(\pm)}(x,y) := |x|^{\frac{1}{2}}\, |y|^{-1}\, \big(\mathcal{F}_1 f\big)(-x, x^{-1}y), \quad x, y \in \mathbb{R}_*^{\pm}, \tag{205}$$

with $\mathcal{F}_1$ denoting the Fourier transform with respect to the first variable. This result — by the essential uniqueness of the inducing kernel of a Hilbert-Schmidt operator (or, more in general, of a Carleman operator; see, for instance, assertion (e) of Theorem 6.13 of [39]) — implies

that $\varsigma_{\mathsf{f}}^{(\pm)}(\cdot,\cdot)$ is the integral kernel associated with the Hilbert-Schmidt operator $\mathfrak{S}_{(\pm)}^{*}\mathsf{f}$, for every $\mathsf{f}\in\mathrm{L}^1(G)\cap\mathrm{L}^2(G)$; hence, we have that

$$\begin{aligned}\|\mathfrak{S}_{(\pm)}^{*}\mathsf{f}\|_{\mathcal{B}_2}^2 = \int_{\mathbb{R}_*^{\pm}}\mathrm{d}x\int_{\mathbb{R}_*^{\pm}}\mathrm{d}y\,\frac{|x|}{y^2}\,\big|\big(\mathcal{F}_1\mathsf{f}\big)(-x,x^{-1}y)\big|^2 &= \int_{\mathbb{R}_*^{\pm}}\mathrm{d}x\int_{\mathbb{R}_*^{+}}\frac{\mathrm{d}r}{r^2}\,\big|\big(\mathcal{F}_1\mathsf{f}\big)(-x,r)\big|^2\\ &\le \int_{\mathbb{R}}\mathrm{d}x\int_{\mathbb{R}_*^{+}}\frac{\mathrm{d}r}{r^2}\,\big|\big(\mathcal{F}_1\mathsf{f}\big)(-x,r)\big|^2\\ &= \int_G|\mathsf{f}(a,r)|^2\,r^{-2}\,\mathrm{d}a\,\mathrm{d}r = \|\mathsf{f}\|_{\mathrm{L}^2}^2.\end{aligned}\tag{206}$$

Of course, what we have found — i.e. $\|\mathfrak{S}_{(\pm)}^{*}\mathsf{f}\|_{\mathcal{B}_2}^2\le\|\mathsf{f}\|_{\mathrm{L}^2}^2$ — is coherent with the fact that the Weyl map $\mathfrak{S}_{(\pm)}^{*}$ is a partial isometry. Now, let f be a *generic* function in $\mathrm{L}^2(G)$ and $\{\mathsf{f}_n\}_{n\in\mathbb{N}}$ a sequence in the linear span $\mathrm{L}^1(G)\cap\mathrm{L}^2(G)$ such that $\lim_{n\to\infty}\|f-\mathsf{f}_n\|_{\mathrm{L}^2}=0$. Then, the sequence $\{\mathfrak{S}_{(\pm)}^{*}\mathsf{f}_n\}_{n\in\mathbb{N}}\subset\mathcal{B}_2(\mathcal{H})$ converges to $\mathfrak{S}_{(\pm)}^{*}f$; equivalently, the sequence $\{\varsigma_{\mathsf{f}_n}^{(\pm)}\}_{n\in\mathbb{N}}$ converges in $\mathrm{L}^2(\mathbb{R}_*^{\pm}\times\mathbb{R}_*^{\pm})$ to the integral kernel of the Hilbert-Schmidt operator $\mathfrak{S}_{(\pm)}^{*}f$, kernel which for the moment is still 'unknown'. But, arguing as in (206), we see that the function $\varsigma_f^{(\pm)}$ belongs to $\mathrm{L}^2(\mathbb{R}_*^{\pm}\times\mathbb{R}_*^{\pm})$ and

$$\|\varsigma_f^{(\pm)}-\varsigma_{\mathsf{f}_n}^{(\pm)}\|_{\mathrm{L}^2(\mathbb{R}_*^{\pm}\times\mathbb{R}_*^{\pm})}^2=\int_{\mathbb{R}_*^{\pm}}\mathrm{d}x\int_{\mathbb{R}_*^{\pm}}\mathrm{d}y\,\frac{|x|}{y^2}\,\big|\big(\mathcal{F}_1(f-\mathsf{f}_n)\big)(-x,x^{-1}y)\big|^2\le\|f-\mathsf{f}_n\|_{\mathrm{L}^2}^2.\tag{207}$$

It follows that the integral kernel of $\mathfrak{S}_{(\pm)}^{*}f$ is $\varsigma_f^{(\pm)}$ for *every* $f\in\mathrm{L}^2(G)$. Moreover, we have that

$$\begin{aligned}\|\mathfrak{S}_{(-)}^{*}f\|_{\mathcal{B}_2}^2+\|\mathfrak{S}_{(+)}^{*}f\|_{\mathcal{B}_2}^2 &= \int_{\mathbb{R}_*^{-}}\mathrm{d}x\int_{\mathbb{R}_*^{+}}\frac{\mathrm{d}r}{r^2}\,\big|\big(\mathcal{F}_1f\big)(-x,r)\big|^2+\int_{\mathbb{R}_*^{+}}\mathrm{d}x\int_{\mathbb{R}_*^{+}}\frac{\mathrm{d}r}{r^2}\,\big|\big(\mathcal{F}_1f\big)(-x,r)\big|^2\\ &= \int_G|f(a,r)|^2\,r^{-2}\,\mathrm{d}a\,\mathrm{d}r=\|f\|_{\mathrm{L}^2}^2,\quad\forall f\in\mathrm{L}^2(G).\end{aligned}\tag{208}$$

Therefore, denoting by $\mathcal{R}_{(\pm)}$ the range of the Wigner map $\mathfrak{S}_{(\pm)}$ (we know that $\mathcal{R}_{(-)}\perp\mathcal{R}_{(+)}$, see Remark 2) — since $\mathcal{R}_{(\pm)}=\mathrm{Ker}\big(\mathfrak{S}_{(\pm)}^{*}\big)^{\perp}$ — the following relation must hold:

$$\mathrm{L}^2(G)=\mathcal{R}_{(-)}\oplus\mathcal{R}_{(+)}.\tag{209}$$

Let us now consider the star products in $\mathrm{L}^2(G)$ associated with the square integrable representations $U^{(-)}$ and $U^{(+)}$. By definition — see (64) — we have

$$f_1\overset{(\pm)}{\star}f_2:=\mathfrak{S}_{(\pm)}\Big(\big(\mathfrak{S}_{(\pm)}^{*}f_1\big)\big(\mathfrak{S}_{(\pm)}^{*}f_2\big)\Big),\quad\forall f_1,f_2\in\mathrm{L}^2(\mathbb{R}\times\mathbb{R}_*^{+},\mu_{\mathrm{L}}).\tag{210}$$

Exploiting the results of Sect. 5 we can provide explicit formulae for these star products. Let $\{\chi_n^{(\pm)}\}_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathrm{L}^2(\mathbb{R}_*^{\pm})$ contained in $\mathrm{Dom}\big(\hat{D}_{(\pm)}^{-2}\big)$; i.e., such that

$$\Big(\mathbb{R}_*^{\pm}\ni x\mapsto|x|\,\chi_n^{(\pm)}(x)\Big)\in\mathrm{L}^2(\mathbb{R}_*^{\pm}).\tag{211}$$

For instance, one can choose the Laguerre functions

$$\chi_n^{(\pm)}\colon\mathbb{R}_*^{\pm}\ni x\mapsto L_{n-1}(|x|)\,\mathrm{e}^{-|x|/2},\quad L_k(x):=\sum_{j=0}^{k}\binom{k}{j}\frac{(-x)^j}{j\,!},\ k=0,1,2,\dots,\tag{212}$$

where, of course, L_k is the Laguerre polynomial of order k. According to the main result of Sect. 5 — see Theorem 2 — we have:

$$f_1\overset{(\pm)}{\star}f_2={}_{\|\cdot\|_{\mathrm{L}^2}}\!\sum_{n\in\mathbb{N}}\int_G\mathrm{d}\mu_{\mathrm{L}}(a,r)\int_G\mathrm{d}\mu_{\mathrm{L}}(a',r')\,\kappa_{(\pm)}\big(\chi_n^{(\pm)};\cdot,\cdot;a,r;a',r'\big)\,f_1(a,r)f_2(a',r'),\tag{213}$$

where the integral kernel $\kappa_{(\pm)}\big(\chi_n^{(\pm)}; \cdot, \cdot; \cdot, \cdot; \cdot, \cdot\big) \colon G \times G \times G \to \mathbb{C}$ is defined by

$$\kappa_{(\pm)}\big(\chi_n^{(\pm)}; a_1, r_1; a_2, r_2; a_3, r_3\big) := \big\langle U^{(\pm)}(a_1, r_1)\, \hat{D}_{(\pm)}^{-1}\, \chi_n^{(\pm)}, U^{(\pm)}(a_2, r_2)\, \hat{D}_{(\pm)}^{-1}\, U^{(\pm)}(a_3, r_3)\, \hat{D}_{(\pm)}^{-1}\, \chi_n^{(\pm)}\big\rangle.$$

Recalling the explicit form of the the Duflo-Moore operators $\hat{D}_{(\pm)}$, we have:

$$\begin{aligned}\kappa_{(\pm)}\big(\chi_n^{(\pm)}; a_1, r_1; a_2, r_2; a_3, r_3\big) &= \frac{r_2\sqrt{r_3}}{r_1}\, \big\langle \hat{D}_{(\pm)}^{-1}\, \chi_n^{(\pm)}, \hat{D}_{(\pm)}^{-2}\, U^{(\pm)}(-(a_1 - a_2 - r_2 a_3)/r_1, r_2 r_3/r_1)\, \chi_n^{(\pm)}\big\rangle \\ &= \left(\frac{r_2}{2\pi r_1}\right)^{\frac{3}{2}} r_3 \int_{\mathbb{R}_*^{\pm}} |x|^{\frac{3}{2}}\, \mathrm{e}^{-\mathrm{i}\,(a_1 - a_2 - r_2 a_3)x/r_1}\, \chi_n^{(\pm)}(x)^*\, \chi_n^{(\pm)}(r_2 r_3 x/r_1)\, \mathrm{d}x \\ &= \left(\frac{r_2}{r_1}\right)^{\frac{3}{2}} \frac{r_3}{2\pi}\, \Lambda_n^{(\pm)}\big((a_1 - a_2 - r_2 a_3)/r_1, r_2 r_3/r_1\big), \end{aligned} \tag{214}$$

where the function $\Lambda_n^{(\pm)} \colon \mathbb{R} \times \mathbb{R}_*^+ \to \mathbb{C}$ is defined by

$$\Lambda_n^{(\pm)}(\alpha, \varrho) := \mathcal{F}\big(|\cdot|^{\frac{3}{2}}\, \breve{\chi}_n^{(\pm)}(\cdot)^*\, \breve{\chi}_n^{(\pm)}(\varrho(\cdot))\big)(\alpha), \quad \alpha \in \mathbb{R},\ \varrho \in \mathbb{R}_*^+, \tag{215}$$

with $\mathcal{F} \colon \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ denoting the Fourier transform $\big(\big(\mathcal{F}\varphi\big)(a) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\,ax} \varphi(x)\, \mathrm{d}x$, for $\varphi \in \mathrm{L}^1(\mathbb{R})\big)$ and $\breve{\chi}_n^{(\pm)} \in \mathrm{L}^2(\mathbb{R})$ the function

$$\breve{\chi}_n^{(\pm)}(x) = \chi_n^{(\pm)}(x), \quad \text{for } x \in \mathbb{R}_*^{\pm}, \qquad \breve{\chi}_n^{(\pm)}(x) = 0, \quad \text{otherwise}; \tag{216}$$

i.e., $\breve{\chi}_n^{(\pm)}$ is the image of $\chi_n^{(\pm)}$ via the natural immersion of $\mathrm{L}^2(\mathbb{R}_*^{\pm})$ into $\mathrm{L}^2(\mathbb{R})$. In conclusion, the triples

$$\mathcal{A}_{(-)} \equiv \big(\mathrm{L}^2(\mathbb{R} \times \mathbb{R}_*^+, \mu_{\mathrm{L}}), \overset{(-)}{\star}, \mathsf{J}\big) \ \text{ and } \ \mathcal{A}_{(+)} \equiv \big(\mathrm{L}^2(\mathbb{R} \times \mathbb{R}_*^+, \mu_{\mathrm{L}}), \overset{(+)}{\star}, \mathsf{J}\big) \tag{217}$$

are H*-algebras. The mutually orthogonal subspaces $\mathcal{R}_{(-)}$ and $\mathcal{R}_{(+)}$ of $\mathrm{L}^2(\mathbb{R} \times \mathbb{R}_*^+, \mu_{\mathrm{L}})$ are, respectively, the canonical and the annihilator ideals in the standard decomposition of the H*-algebra $\mathcal{A}_{(-)}$, while they are, respectively, the annihilator and the canonical ideals for $\mathcal{A}_{(+)}$. It is clear that one may endow $\mathrm{L}^2(\mathbb{R} \times \mathbb{R}_*^+, \mu_{\mathrm{L}})$ with the structure of a *proper* H*-algebra by considering the star product

$$f_1 \star f_2 := \big(f_1 \overset{(-)}{\star} f_2\big) + \big(f_1 \overset{(+)}{\star} f_2\big). \tag{218}$$

Let us now clarify the link with the standard wavelet transform. To this aim, let us consider the unitary representation $\widetilde{U} \colon G \to \mathcal{U}(\mathrm{L}^2(\mathbb{R}))$ defined as follows. Taking into account the orthogonal sum decomposition $\mathrm{L}^2(\mathbb{R}) = \mathrm{L}^2(\mathbb{R}_*^-) \oplus \mathrm{L}^2(\mathbb{R}_*^+)$, we can consider the representation $U^{(-)} \oplus U^{(+)}$ of G in $\mathrm{L}^2(\mathbb{R})$; then, we set

$$\widetilde{U}(a, r) := \mathcal{F}\Big(\big(U^{(-)} \oplus U^{(+)}\big)(a, r)\Big)\mathcal{F}^*, \quad \forall\, (a, r) \in \mathbb{R} \rtimes \mathbb{R}_*^+. \tag{219}$$

For every $\psi \in \mathrm{L}^2(\mathbb{R})$, we have:

$$\psi_{a,r}(a') \equiv \big(\widetilde{U}(a, r)\, \psi\big)(a') = r^{-\frac{1}{2}}\, \psi((a' - a)/r), \quad a, a' \in \mathbb{R},\ r \in \mathbb{R}_*^+. \tag{220}$$

Observe that this is the typical dependence on the translation and dilation parameters of a 'wavelet frame' (see [26]; note that the symbols that we use here for these parameters are non-standard). However, a function $\psi \in \mathrm{L}^2(\mathbb{R})$, in order to be a 'good mother wavelet' — i.e. in order to verify the the orthogonality relations

$$\int_G \langle \phi, \psi_{a,r}\rangle \langle \psi_{a,r}, \phi\rangle\, \mathrm{d}\mu_{\mathrm{L}}(a, r) = \langle \phi, \phi\rangle, \quad \forall\, \phi \in \mathrm{L}^2(\mathbb{R}) \tag{221}$$

— has to satisfy suitable conditions. Indeed, as the reader will easily understand, one has to require that the following conditions hold:

1. the projection onto $\mathrm{L}^2(\mathbb{R}^\pm_*)$ (regarded as a subspace of $\mathrm{L}^2(\mathbb{R})$) of the Fourier transform of ψ belongs to $\mathrm{Dom}\big(\hat{D}_{(\pm)}\big)$, i.e.

$$\Big(\mathbb{R}^\pm_* \ni x \mapsto |x|^{-1}\left|\big(\mathcal{F}\,\psi\big)(x)\right|^2\Big) \in \mathrm{L}^1(\mathbb{R}^\pm_*); \tag{222}$$

2. denoting by $\varepsilon_{\mathbb{R}^\pm_*}$ the characteristic function of the subset $\mathbb{R}^\pm_*$ of $\mathbb{R}$,[11] the vectors

$$\hat{D}_{(-)}\left(\varepsilon_{\mathbb{R}^-_*}(\mathcal{F}\,\psi)\right) \in \mathrm{L}^2(\mathbb{R}^-_*) \;\text{ and }\; \hat{D}_{(+)}\left(\varepsilon_{\mathbb{R}^+_*}(\mathcal{F}\,\psi)\right) \in \mathrm{L}^2(\mathbb{R}^+_*)$$

are both normalized, i.e.

$$2\pi\int_{\mathbb{R}^-_*} |x|^{-1}\left|\big(\mathcal{F}\,\psi\big)(x)\right|^2 \mathrm{d}x = 2\pi\int_{\mathbb{R}^+_*} |x|^{-1}\left|\big(\mathcal{F}\,\psi\big)(x)\right|^2 \mathrm{d}x = 1. \tag{223}$$

One can obtain a variant of the scheme analyzed above by allowing both positive and *negative* dilations; this variant is widely exploited in wavelet analysis, see [26]. It amounts to considering the semidirect product $\mathbb{R}\rtimes\mathbb{R}_*$, with $\mathbb{R}_*$ denoting the group of nonzero real numbers (with respect to multiplication). This semidirect product group admits a single square integrable irreducible representation, up to unitary equivalence; namely, the unitary representation $U\colon G\to\mathcal{U}(\mathrm{L}^2(\mathbb{R}))$ defined by

$$\big(U(a,r)\,\varphi\big)(x) := |r|^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}\,ax}\varphi(rx), \quad a\in\mathbb{R},\; r\in\mathbb{R}_*,\; x\in\mathbb{R},\; \varphi\in\mathrm{L}^2(\mathbb{R}). \tag{224}$$

Of course, one can repeat for $\mathbb{R}\rtimes\mathbb{R}_*$ the same analysis performed for the group $\mathbb{R}\rtimes\mathbb{R}^+_*$. We leave this analysis as an exercise for the reader.

7 Conclusions, final remarks and perspectives

In this paper we have considered star products from a purely group-theoretical point of view. In particular, we have not assumed to deal with Lie groups, but, in general, with locally compact topological groups. Therefore, our treatment allows us to include in a unified framework, for instance, all the finite groups (in the paper regarded as compact groups). This feature is certainly appealing in view of the increasing interest in realizing quantum mechanics on discrete spaces (see [40] and references therein). We think, in particular, that applying our results to a formulation of quantum mechanics on finite groups would be extremely interesting.

Let us briefly review the main points of our work.

We have first recalled — see Sect. 3 — that with a square integrable (in general, projective) representation $U\colon G\to\mathcal{U}(\mathcal{H})$ of a locally compact group G are naturally associated a dequantization (Wigner) map $\mathfrak{S}_U$, which is an isometry, and its adjoint, the quantization (Weyl) map $\mathfrak{S}_U^*$. The standard Wigner and Weyl maps are recovered in the case where the group under consideration is the group of translations on phase space, up to a (symplectic) Fourier transform. We stress that this Fourier transform does not play any — mathematically or conceptually — relevant role; essentially, it allows to obtain the usual quantization rule for the functions of position and momentum.

Next, in Sect. 4, we have observed that by means of the quantization and dequantization maps associated with the representation U one can define a star product of functions enjoying remarkable properties. Endowed with this product and with a suitable involution, the Hilbert space $\mathrm{L}^2(G)$ becomes a H*-algebra $\mathcal{A}_U$, and — regarding G as a 'symmetry group' of a quantum

[11] Observe that the orthogonal projection of $\mathrm{L}^2(\mathbb{R})$ onto $\mathrm{L}^2(\mathbb{R}^\pm_*)$ is just the multiplication operator by $\varepsilon_{\mathbb{R}^\pm_*}$.

system — the star product is, by construction, equivariant with respect to the natural action of G in $\mathcal{A}_U$, i.e. the action with which the standard symmetry action of G on states or observables in the Hilbert space $\mathcal{H}$ is intertwined via the Wigner map. Observe that the star product associated with U is such that the canonical ideal of $\mathcal{A}_U$ — ideal which coincides with the range $\mathcal{R}_U$ of $\mathfrak{S}_U$ — is a *simple* H*-algebra (see [32, 33]), isomorphic to $\mathcal{B}_2(\mathcal{H})$. It is clear that the algebra $\mathcal{A}_U$ is commutative if and only if $\dim(\mathcal{H}) = 1$ (in this case, the square-integrability of U forces the group G to be compact). Observe moreover that, in the case where G admits various (unitarily) inequivalent *unitary* representations, one can define more general star products by forming suitable 'orthogonal sums' of 'simple' star products; see, e.g., formula (218). In Sect. 4, we have also considered an interesting deformation of the star product associated with U, namely, the $\hat{K}$-deformed star product, and studied its main properties. We will consider applications of this deformed product elsewhere.

At this point, our main task has been to derive explicit formulae for the previously defined star products. This task has been accomplished in Sect. 5. We have shown that for every orthonormal basis contained in the domain of the positive selfadjoint operator $\hat{D}_U^{-2}$ (with $\hat{D}_U$ denoting the Duflo-Moore operator associated with U) one has a realization of the star product, see Theorem 2; more generally — see Theorem 3 — for every suitable right approximate identity in the H*-algebra $\mathcal{B}_2(\mathcal{H})$ one can provide a realization of the $\hat{K}$-deformed star product. In the case where the group G is unimodular, the star product of two functions *belonging to the range of the Wigner map* $\mathfrak{S}_U$ assumes the particularly simple form of a 'twisted convolution', which reduces to the standard convolution if U is a unitary representation. It is interesting to note, incidentally, that it is the Banach space $\mathrm{L}^1(G)$ which is usually endowed with the structure of a Banach $*$-algebra by means of convolution [22], while in $\mathrm{L}^2(G)$ the convolution product is, in general, an 'ill-posed' operation. Namely, if the convolution product exists and belongs to $\mathrm{L}^2(G)$ *for all pairs of functions* in $\mathrm{L}^2(G)$, then the group G must be compact (recall, however, that by Hölder's inequality the convolution of any pair of functions in $\mathrm{L}^2(G)$ does exist, for G unimodular). This is a particular case ($p = 2$) of the classical 'L^p-conjecture' ($p > 1$), which has been finally proved (in its general form) in 1990 by Saeki [41]. Therefore, the *whole* vector space $\mathrm{L}^2(G)$ can be endowed with the structure of an algebra by means of the convolution product if and only if G is compact.

Consider, now, the specific case where the group G is compact. In this case, one obtains a nice decomposition formula for the convolution in $\mathrm{L}^2(G)$ in terms of the star products associated with a realization of the unitary dual $\check{G}$ of G; see Corollary 2. The Hilbert space $\mathrm{L}^2(G)$, endowed with the convolution product and with the involution (138), is a H*-algebra which we denote by $\mathcal{L}(G)$. The orthogonal sum decomposition (36) — complemented by formula (137) — can be regarded as the decomposition into minimal closed (two-sided) ideals of $\mathcal{L}(G)$ prescribed by the 'second Wedderburn structure theorem for H*-algebras' [32, 33]. Any of these ideals — say $\mathcal{R}_U = \mathrm{L}^2(G)_{[U]}$ — is a simple finite-dimensional H*-algebra which is embedded, in a natural way, in the H*-algebra $\mathcal{A}_U$ determined by the star product (151) and by the involution (138); precisely, as already observed, $\mathcal{R}_U$ is the canonical ideal of $\mathcal{A}_U$. It is actually the interest in the algebra $\mathcal{L}(G)$ that motivated Ambrose's study of H*-algebras [32]. On our opinion, the formalism of star products provides a concrete and conceptually clear framework for Ambrose's ideas.

It is worth noting that — differently from the quantization-dequantization scheme which has been recently developed in ref. [17] — in the 'Weyl-Wigner approach' that is considered in the present contribution there is no canonical way for representing a *generic* quantum observable as a suitable 'phase space function' since, for $\mathcal{H}$ infinite-dimensional, $\mathcal{B}_2(\mathcal{H}) \subsetneq \mathcal{B}(\mathcal{H})$ (in the case of the standard Weyl quantization, this problem has been studied, for instance, in [42]). This feature, of course, reflects into the fact that there is no standard way for representing

within the framework considered here the product of a *generic* quantum observable by a state as a star product of functions. However, we believe that suitably extending the domain of the first argument of the star product — this time *defined* as the r.h.s. of (127) — from $\mathrm{L}^2(G)$ to some larger space of functions (or distributions), and, possibly, restricting the domain of the second argument, one should be able to generalize the results obtained in the paper. This interesting topic will be the object of further investigation.

One can, in principle, elaborate several examples of star products defined along the lines traced in the present paper that are potentially relevant for applications. In addition to the case of compact groups, for all groups admitting square integrable projective representations it is possible to define star products of functions. In Sect. 6 we have considered the significant examples of the group of translations on phase space and of the affine group, but, of course, several other examples would deserve attention. As an example, we mention the group $\mathrm{SL}(2,\mathbb{R})$. According to classical results due to Bargmann [43], this group admits a (infinite) countable set of mutually inequivalent square integrable unitary representations — the 'discrete series' — with carrier Hilbert spaces consisting of suitable holomorphic functions on the upper half plane.

A wide class of groups with important applications in physics and related research areas (in particular, signal analysis) is formed by the semidirect products with an abelian normal factor. For these groups square integrable representations can be suitably characterized, see [38], and examples of such groups, admitting square integrable representations and having remarkable applications, can be found in refs. [18, 19]. From the point of view of signal analysis, the image through the Weyl map of a function in $\mathrm{L}^2(G)$ can be regarded as a *localization operator* of a different kind with respect to the localization operators usually considered in wavelet and Gabor analysis [26]. Thus, the star product provides a way for characterizing the product of two localization operators. Possible applications of our results to signal analysis is a further topic that we plan to investigate in the future.

References

[1] C.K. Zachos, D.B. Fairlie and T.L. Curtright eds., *Quantum Mechanics in Phase Space*, World Scientific (2005).

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, "Deformation theory and quantization. I. Deformations of symplectic structures.", *Ann. Phys.* **111** (1978), 61-110; F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, "Deformation theory and quantization. II. Physical applications.", *ibidem* **111** (1978), 111-151.

[3] M. Cahen, S. Gutt, "Regular $*$-representations of Lie algebras", *Lett. Math. Phys.* **6** (1982), 395-404.

[4] S. Gutt, "An explicit $*$-product on the cotangent bundle of a Lie group", *Lett. Math. Phys.* **7** (1983), 249-258.

[5] M. De Wilde, P. Lecompte, "Existence of star-products and of formal deformations of the Poisson-Lie algebra of arbirary symplectic manifolds", *Lett. Math. Phys.* **7** (1983), 487-496.

[6] F.A. Berezin, "General concept of quantization", *Comm. Math. Phys.* **40** (1975), 153-174.

[7] G.G. Emch, "Mathematical and Conceptual Foundations of 20th-Century Physics", North-Holland (1984).

[8] E. Wigner, “On the quantum correction for thermodynamic equilibrium”, *Phys. Rev.* **40** (1932), 749-759.

[9] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover (1950).

[10] J.C. Várilly, J.M. Gracia-Bondía, “The Moyal representation for spin”, *Ann. Phys.* **190** (1989), 107-148.

[11] M. Gadella, M.M. Martín, L.M. Nieto, M.A. del Olmo, “The Stratonovich-Weyl correspondence for one-dimensional kinematical groups”, J. Math. Phys. **32** (1991), 1182-1192.

[12] J.M. Gracia-Bondía, J.C. Várilly, H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, Boston (2001).

[13] O.V. Man'ko, V.I. Man'ko, G. Marmo, “Alternative commutation relations, star products and tomography”, *J. Phys. A: Math. Gen.* **35** (2002), 699-719.

[14] V.I. Man'ko, G. Marmo, P. Vitale, “Phase space distributions and a duality symmetry for star products”, *Phys. Lett. A* **334** (2005), 1-11.

[15] O.V. Man'ko, V.I. Man'ko, G. Marmo, P. Vitale, “Star products, duality and double Lie algebras”, *Phys. Lett. A* **360** (2007), 522-532.

[16] P. Aniello, A. Ibort, V.I. Man'ko, G. Marmo, “Remarks on the star product of functions on finite and compact groups”, *Phys. Lett. A* **373** (2009), 401-408.

[17] P. Aniello, V.I. Man'ko, G. Marmo, “Frame transforms, star products and quantum mechanics on phase space”, *J. Phys. A: Math. Theor.* **41** (2008), 285304.

[18] S.T. Ali, J.P. Antoine, J.P. Gazeau, U.A. Mueller, “Coherent states and their generalizations: a mathematical overview”, *Rev. Math. Phys.* **7** (1995), 1013-1104.

[19] S.T. Ali, J.P. Antoine, J.P. Gazeau, *Coherent States, Wavelets and Their Generalizations*, Springer-Verlag (2000).

[20] A. Grossmann, G. Loupias, E.M. Stein, “An algebra of pseudo-differential operators and quantum mechanics in phase space”, *Ann. Inst. Fourier* **18** (1968), 343-368.

[21] V.S. Varadarajan, *Geometry of Quantum Theory*, second edition, Springer (1985).

[22] G.B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press (1995).

[23] P. Aniello, “Square integrable projective representations and square integrable representations modulo a relatively central subgroup”, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006), 233-267.

[24] M. Duflo, C.C. Moore, “On the regular representation of a nonunimodular locally compact group”, *J. Funct. Anal.* **21** (1976), 209-243.

[25] A. Grossmann, J. Morlet, T. Paul, “Integral transforms associated to square integrable representations I. General results.”, *J. Math. Phys.* **26** (1985), 2473-2479; A. Grossmann, J. Morlet, T. Paul, “Integral transforms associated to square integrable representations II. Examples”, *Ann. Inst. H. Poincaré* **45** (1986), 293-309.

[26] I. Daubechies, *Ten Lectures on Wavelets*, SIAM (1992).

[27] B. Simon, *Representations of Finite and Compact Groups*, American Mathematical Society (1996).

[28] A. Perelomov, *Generalized Coherent States and Their Applications*, Springer-Verlag (1986).

[29] J.R. Klauder, E.C.G. Sudarshan, *Fundamentals of Quantum Optics*, W.A. Benjamin (1968).

[30] M. Reed, B Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press (1972).

[31] I.E. Segal, "The two-sided regular representation of a unimodular locally compact group", *Ann. Math.* **51** (1950), 293-298.

[32] W. Ambrose, "Structure theorems for a certain class of Banach algebras", *Trans. Amer. Math Soc.* **57** (1945), 364-386.

[33] C.E. Rickart, "General Theory of Banach Algebras", Van Nostrand (1960).

[34] G. Esposito, G. Marmo, G. Sudarshan, *From Classical to Quantum Mechanics*, Cambridge University Press (2004).

[35] G.B. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press (1989).

[36] J.C.T. Pool, "Mathematical aspects of Weyl correspondence", *J. Math. Phys.* **7** (1966), 66-76.

[37] E.W. Aslaksen, J.R. Klauder, "Continuous representation theory using the affine group", *J. Math. Phys.* **10** (1969), 2267-2275.

[38] P. Aniello, G. Cassinelli, E. De Vito, A. Levrero, "Square-integrability of induced representations of semidirect products", *Rev. Math. Phys.* **10** (1998), 301-313.

[39] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer-Verlag (1980).

[40] K.S. Gibbons, M.J. Hoffmann, W.K. Wootters, "Discrete phase space based on finite fields", *Phys. Rev. A* **70** (2004), 062101.

[41] S. Saeki, "The L^p-conjecture and Young's inequality", *Illinois J. Math.* **34** (1990), 615-627.

[42] I. Daubechies, "On the distributions corresponding to bounded operators in the Weyl quantization", *Commun. Math. Phys.* **75** (1980), 229-238.

[43] V. Bargmann, "Irreducible representations of the Lorentz group", *Ann. of Math.* **48** (1947), 568-640.